12

Special Relativity

12.1 Inertial frames and Lorentz transformations

An **inertial reference frame** is a system of coordinates in which free particles move in straight lines at constant speeds. Our spacetime has one time dimension $x^0 = ct$ and three space dimensions $\boldsymbol{x}$. Its physical points are labeled by four coordinates, $p = (x^0, x^1, x^2, x^3)$. The quadratic separation between two infinitesimally separated points p and $p + dp$ whose coordinates differ by dx^0, dx^1, dx^2, dx^3 is

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^0)^2. \tag{12.1}$$

In the absence of gravity, ds^2 is the **physical** quadratic separation between the points p and $p + dp$. Since it is physical, it must not change when we change coordinates. Changes of coordinates $x \to x'$ that leave ds^2 invariant are called **Lorentz transformations**.

If we adopt the **summation convention** in which an index is summed from 0 to 3 if it occurs both raised and lowered in the same monomial, then we can write a Lorentz transformation as

$$x'^i = \sum_{k=0}^{3} L^i{}_k \, x^k = L^i{}_k \, x^k. \tag{12.2}$$

Lorentz transformations change coordinate differences dx^k to $dx'^i = L^i{}_k \, dx^k$.

The metric of flat spacetime $\eta_{ik} = \eta^{ik}$ is the 4×4 matrix

$$(\eta_{ik}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\eta^{ik}). \tag{12.3}$$

It is its own inverse because $\eta_{ik} \, \eta^{k\ell} = \delta_i^\ell$.

In terms of the metric η, the Minkowski **inner product** (example 1.21) between two 4-vectors x^i and y^k is

$$(x, y) \equiv x \cdot y = \boldsymbol{x} \cdot \boldsymbol{y} - x^0 y^0 = x^i \, \eta_{ik} \, y^k. \tag{12.4}$$

The quadratic distance (12.1), for example, is the inner product

$$ds^2 = (dx, dx) = d\boldsymbol{x} \cdot d\boldsymbol{x} - (dx^0)^2 = dx^i \, \eta_{ik} \, dx^k. \tag{12.5}$$

Lorentz transformations are defined as linear maps that preserve Minkowski inner products

$$x'^i \eta_{ik} y'^k = L^i{}_\ell x^\ell \, \eta_{ik} \, L^k{}_j y^j = x^\ell \eta_{\ell j} y^j. \tag{12.6}$$

Since they leave Minkowski inner products unchanged, Lorentz transformations form a group as discussed in sections 11.1 and 11.39.

By differentiating the defining equation (12.6) with respect to dx^ℓ and dy^j, one may show (exercise 12.2) that the matrix L obeys the equation

$$L^i{}_\ell \, \eta_{ik} \, L^k{}_j = \eta_{\ell j}. \tag{12.7}$$

Multiplying this equation by $\eta^{m\ell}$, we get

$$\eta^{m\ell} L^i{}_\ell \, \eta_{ik} \, L^k{}_j = \eta^{m\ell} \eta_{\ell j} = \delta^m_j \tag{12.8}$$

which tells us that the inverse of the Lorentz transformation $L^k{}_j$ is

$$L^{-1m}{}_k = \eta^{m\ell} L^i{}_\ell \, \eta_{ik}. \tag{12.9}$$

If $x' = Lx$ as in (12.2), then the inverse L^{-1} of L takes x' back to x

$$x^i = L^{-1i}{}_k x'^k = L^{-1i}{}_k L^k{}_\ell x^\ell = \delta^i_\ell \, x^\ell = x^i. \tag{12.10}$$

Thus derivatives transform as

$$\frac{\partial}{\partial x'^k} = \frac{\partial x^i}{\partial x'^k} \frac{\partial}{\partial x^i} = L^{-1i}{}_k \frac{\partial}{\partial x^\ell}. \tag{12.11}$$

If A^i is any 4-vector with an upper index i, then the same 4-vector with a lower index is

$$A_i = \eta_{ik} \, A^k. \tag{12.12}$$

The 4-vector A_i is the same as A^i except that $A_0 = -A^0$ has a hidden minus sign. And if B_i is any 4-vector with a lower index, then the same 4-vector with an upper index is

$$B^k = \eta^{ik} \, B_i \tag{12.13}$$

which is the same as B_k except that $B^0 = -B_0$ has no hidden minus sign. These rules are consistent because η is its own inverse $\eta^{ki}\,\eta_{ij} = \delta^k_j$ and so

$$B^k = \eta^{ki}\,B_i = \eta^{ki}\,\eta_{ij}\,B^j = \delta^k_j\,B^j = B^k. \tag{12.14}$$

Thus another way of writing the inverse Lorentz transformation (12.9 is

$$L^{-1m}{}_k = \eta^{m\ell} L^i{}_\ell\,\eta_{ik} = L_k{}^m. \tag{12.15}$$

The **contraction** $A_j B^j$ of any two 4-vectors is invariant because Lorentz transformations obey (12.7)

$$A'_i B'^i = \eta_{ik} A'^k B'^i = \eta_{ik} L^k{}_\ell A^\ell L^i{}_j B^j = \eta_{\ell j} A^\ell B^j = A_j B^j. \tag{12.16}$$

Thus 4-divergences are invariant $\partial'_i A'^i = \partial_j A^j$.

Example 12.1 (Lorentz transformations) If we change coordinates to

$$\begin{aligned} x'^0 &= \cosh\theta\,x^0 + \sinh\theta\,x^1 \\ x'^1 &= \sinh\theta\,x^0 + \cosh\theta\,x^1 \\ x'^2 &= x^2 \quad \text{and} \quad x'^3 = x^3, \end{aligned} \tag{12.17}$$

then the Lorentz transformations L (exercise 12.3) and L^{-1} are

$$L = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L^{-1} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12.18}$$

which are boosts in the x-direction to speeds $\pm v/c = \pm\tanh\theta$ and **rapidity** θ. One may check (exercise 12.4) that $L^i{}_\ell\,\eta_{ik}\,L^k{}_j = \eta_{\ell j}$. In the new coordinates, the point $p = (x^0, x^1, x^2, x^3)$ is

$$p = (\cosh\theta\,x'^0 - \sinh\theta\,x'^1, \cosh\theta\,x'^1 - \sinh\theta\,x'^0, x'^2, x'^3). \tag{12.19}$$

□

Example 12.2 (Spacelike points) Points p and q with

$$\Delta s^2 = (p-q)\cdot(p-q) = (\boldsymbol{p}-\boldsymbol{q})^2 - (p^0 - q^0)^2 > 0 \tag{12.20}$$

are said to be **spacelike**; their separation Δs is the **proper distance** between the points.

Spacelike events occur at the same time in some Lorentz frames. Let the coordinates of p and q be $(0,0,0,0)$ and $(ct, L, 0, 0)$ with $|ct/L| < 1$ so that $(p-q)^2 > 0$. The Lorentz transformation (12.18) leaves p unchanged but takes q to $q' = (ct\cosh\theta + L\sinh\theta, ct\sinh\theta + L\cosh\theta, 0, 0)$. So both $p' = p$ and q' occur at the time $t' = 0$ if $v/c = \tanh\theta = -ct/L$. □

Example 12.3 (Timelike points) Points p and q with

$$\Delta s^2 = (p-q)\cdot(p-q) = (\boldsymbol{p}-\boldsymbol{q})^2 - (p^0-q^0)^2 < 0 \tag{12.21}$$

are said to be **timelike**; their separation $\sqrt{-\Delta s^2}$ is the proper time between the events. Timelike events occur at the same place in some Lorentz frames.

We can use the same coordinates as in the previous example (12.2) but with $|ct/L| > 1$ so that $(p-q)^2 < 0$. The Lorentz transformation (12.18) leaves p unchanged but takes q to $(ct\cosh\theta + L\sinh\theta, ct\sinh\theta + L\cosh\theta, 0, 0)$. So if $v/c = \tanh\theta = -L/(ct)$, then p' and q' occur at the same place $\mathbf{0}$. □

Example 12.4 (Lightlike points) Points p and q whose quadratic separation vanishes

$$\Delta s^2 = (p-q)\cdot(p-q) = (\boldsymbol{p}-\boldsymbol{q})^2 - (p^0-q^0)^2 = 0 \tag{12.22}$$

can be connected by a light ray and are said to be **lightlike**. □

In special relativity, the spacetime coordinates $x^0 = ct$ and $(x^1, x^2, x^3) = \boldsymbol{x}$ of a point have upper indexes and transform contravariantly $x'^k = L^k{}_\ell x^\ell$. Spacetime derivatives $\partial_0 = \partial/\partial x^0$ and $\nabla = (\partial_1, \partial_2, \partial_3)$ have lower indexes and transform in accordance with $x^\ell = L^{-1\ell}{}_j x'^j$ and That is, the since the inverse of Lorentz transformation by (12.9) is , derivatives

12.2 Special relativity

The spacetime of special relativity is flat, four-dimensional Minkowski space. In the absence of gravity, the inner product $(p-q)\cdot(p-q)$

$$(p-q)\cdot(p-q) = (\boldsymbol{p}-\boldsymbol{q})^2 - (p^0-q^0)^2 = (p-q)^i\,\eta_{ik}\,(p-q)^k \tag{12.23}$$

is physical and the same in all Lorentz frames. If the points p and q are close neighbors with coordinates $x^i + dx^i$ for p and x^i for q, then that invariant inner product is $ds^2 = dx^i\,\eta_{ij}\,dx^j = \boldsymbol{dx}^2 - (dx^0)^2$.

If the points p and q are on the trajectory of a massive particle moving at velocity $\boldsymbol{v}$, then this invariant quadratic separation

$$ds^2 = \boldsymbol{dx}^2 - c^2dt^2 = \left(\boldsymbol{v}^2 - c^2\right)dt^2 \tag{12.24}$$

is negative since $v < c$. The time in the rest frame of the particle is the **proper time** τ, and

$$d\tau^2 = -\,ds^2/c^2 = \left(1 - \boldsymbol{v}^2/c^2\right)dt^2. \tag{12.25}$$

A particle of mass zero moves at the speed of light, so its element $d\tau$ of proper time is zero. But for a particle of mass $m > 0$ moving at speed v, the element of proper time $d\tau$ is smaller than the corresponding element of laboratory time dt by the factor $\sqrt{1 - \boldsymbol{v}^2/c^2}$. The proper time is the time in the rest frame of the particle where $\boldsymbol{v} = 0$. So if $T(0)$ is the lifetime of a particle at rest, then the apparent lifetime $T(v)$ when the particle is moving at speed v is

$$T(v) = dt = \frac{d\tau}{\sqrt{1 - v^2/c^2}} = \frac{T(0)}{\sqrt{1 - v^2/c^2}} \tag{12.26}$$

which is longer than $T(0)$ since $1 - v^2/c^2 \le 1$, an effect known as **time dilation**.

Example 12.5 (Time dilation in muon decay) A muon at rest has a mean life of $T(0) = 2.2 \times 10^{-6}$ seconds. Cosmic rays hitting nitrogen and oxygen nuclei make pions high in the Earth's atmosphere. The pions rapidly decay into muons in 2.6×10^{-8} s. A muon moving at the speed of light from 10 km takes at least $t = 10\,\text{km}/300,000\,(\text{km/sec}) = 3.3 \times 10^{-5}$ s to hit the ground. Were it not for time dilation, the probability P of such a muon reaching the ground as a muon would be

$$P = e^{-t/T(0)} = \exp(-33/2.2) = e^{-15} = 2.6 \times 10^{-7}. \tag{12.27}$$

The mass of a muon is 105.66 MeV. So a muon of energy $E = 749$ MeV has by (12.35) a time-dilation factor of

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} = \frac{749}{105.7} = 7.089 = \frac{1}{\sqrt{1 - (0.99)^2}}. \tag{12.28}$$

So a muon moving at a speed of $v = 0.99\,c$ has an apparent mean life $T(v)$ given by equation (12.26) as

$$T(v) = \frac{E}{mc^2}\,T(0) = \frac{T(0)}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6}\,\text{s}}{\sqrt{1 - (0.99)^2}} = 1.6 \times 10^{-5}\,\text{s}. \tag{12.29}$$

The probability of survival with time dilation is

$$P = e^{-t/T(v)} = \exp(-33/16) = 0.12 \tag{12.30}$$

so that 12% survive. Time dilation increases the chance of survival by a factor of 460,000—no small effect. □

12.3 Kinematics

From the scalar $d\tau = \sqrt{1 - v^2/c^2}\, dt$, and the contravariant vector dx^i, we can make the 4-vector

$$u^i = \frac{dx^i}{d\tau} = \frac{dt}{d\tau} \left(\frac{dx^0}{dt}, \frac{d\boldsymbol{x}}{dt} \right) = \frac{1}{\sqrt{1 - v^2/c^2}} \, (c, \boldsymbol{v}) \tag{12.31}$$

in which $u^0 = c\, dt/d\tau = c/\sqrt{1 - v^2/c^2}$ and $\boldsymbol{u} = u^0\, \boldsymbol{v}/c$. The inner product $u_i u^i$ is a constant $-c^2$ because

$$u_i u^i = \boldsymbol{u} \cdot \boldsymbol{u} - (u^0)^2 = \frac{v^2 - c^2}{1 - v^2/c^2} = -c^2. \tag{12.32}$$

The product mu^i is the **energy-momentum 4-vector** p^i

$$\begin{aligned} p^i = m\, u^i = m\, \frac{dx^i}{d\tau} &= m\, \frac{dt}{d\tau}\, \frac{dx^i}{dt} = \frac{m}{\sqrt{1 - v^2/c^2}}\, \frac{dx^i}{dt} \\ &= \frac{m}{\sqrt{1 - v^2/c^2}}\, (c, \boldsymbol{v}) = \left(\frac{E}{c}, \boldsymbol{p} \right). \end{aligned} \tag{12.33}$$

Its invariant inner product is a constant characteristic of the particle and proportional to the square of its mass because $u_i u^i = -c^2$ (12.32)

$$c^2\, p^i\, p_i = mc\, u^i\, mc\, u_i = -E^2 + c^2\, \boldsymbol{p}^{\,2} = -m^2\, c^4. \tag{12.34}$$

Note that the time-dilation factor is the ratio of the energy of a particle to its rest energy

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{E}{mc^2} \tag{12.35}$$

and the velocity of the particle is its momentum divided by its equivalent mass E/c^2

$$\boldsymbol{v} = \frac{\boldsymbol{p}}{E/c^2}. \tag{12.36}$$

The analog of $\boldsymbol{F} = m\, \boldsymbol{a}$ is

$$m\, \frac{d^2 x^i}{d\tau^2} = m\, \frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i \tag{12.37}$$

in which $p^0 = E/c$, and f^i is a 4-vector force. The **4-acceleration**

$$a^i = \frac{d^2 x^i}{d\tau^2} = \frac{du^i}{d\tau} \tag{12.38}$$

is perpendicular to the 4-velocity because $u_i u^i = -c^2$ (12.32)

$$0 = u_i a^i = u_i \frac{du^i}{d\tau} = \frac{1}{2} \frac{d(u_i u^i)}{d\tau}. \tag{12.39}$$

Example 12.6 (Time dilation and proper time) In the frame of a laboratory, a particle of mass m with 4-momentum $p^i_{lab} = (E/c, p, 0, 0)$ travels a distance L in a time t for a 4-vector displacement of $x^i_{lab} = (ct, L, 0, 0)$. In its own rest frame, the particle's 4-momentum and 4-displacement are $p^i_{rest} = (mc, 0, 0, 0)$ and $x^i_{rest} = (c\tau, 0, 0, 0)$. Since the Minkowski inner product of two 4-vectors is Lorentz invariant, we have

$$\left(p^i x_i\right)_{rest} = \left(p^i x_i\right)_{lab} \text{ or } pL - Et = -mc^2\tau = -mc^2 t\sqrt{1 - v^2/c^2}. \qquad (12.40)$$

So a massive particle's phase $\exp(ip^i x_i/\hbar)$ is $\exp(-imc^2\tau/\hbar)$. □

Example 12.7 ($p + p \to 3p + \bar{p}$) Conservation of the energy-momentum 4-vector gives $p + p_0 = 3p' + \bar{p}'$. We set $c = 1$ and use this equality in the invariant form $(p + p_0)^2 = (3p' + \bar{p}')^2$. We compute $(p + p_0)^2 = p^2 + p_0^2 + 2p \cdot p_0 = -2m_p^2 + 2p \cdot p_0$ in the laboratory frame in which $p_0 = (m, \mathbf{0})$. Thus $(p+p_0)^2 = -2m_p^2 - 2E_p m_p$. We compute $(3p'+\bar{p}')^2$ in the frame in which each of the three protons and the antiproton has zero spatial momentum. There $(3p' + \bar{p}')^2 = (4m, \mathbf{0})^2 = -16m_p^2$. We get $E_p = 7m_p$ of which $6m_p = 5.63$ GeV is the threshold kinetic energy of the proton. In 1955, when the group led by Owen Chamberlain and Emilio Segrè discovered the antiproton, the nominal maximum energy of the protons in the Bevatron was 6.2 GeV. □

12.4 Electrodynamics

In electrodynamics and in MKSA (SI) units, the three-dimensional vector potential $\boldsymbol{A}$ and the scalar potential ϕ form the 4-vector contravariant A^i and covariant A_i potentials

$$A^i = (\phi/c, \boldsymbol{A}) \quad \text{and} \quad A_i = (-\phi/c, \boldsymbol{A})\,. \qquad (12.41)$$

The **magnetic induction** is the curl of the vector potential

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad \text{or} \quad B_i = \epsilon_{ijk}\partial_j A_k \qquad (12.42)$$

in which $\partial_j = \partial/\partial x^j$, the sum over the repeated indices j and k runs from 1 to 3, and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$. The electric field is

$$\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}} \quad \text{or} \quad E_i = c\left(\frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0}\right) = -\frac{\partial \phi}{\partial x^i} - \frac{\partial A_i}{\partial t} \qquad (12.43)$$

where $x^0 = ct$.

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are unchanged by the **gauge transformation**

$$A_0' = A_0 + \partial_0 \lambda \quad \text{and} \quad \boldsymbol{A}' = \boldsymbol{A} + \nabla \lambda \tag{12.44}$$

(exercise 12.15).

Example 12.8 (Potentials of static electric and magnetic fields) In the Coulomb or radiation gauge, a static point charge q at the origin has the potentials $A_0 = q/(4\pi\epsilon_0 c r)$ and $\boldsymbol{A} = 0$. Its the electric field is

$$\boldsymbol{E} = \frac{q\,\boldsymbol{r}}{4\pi\epsilon_0 r^3}. \tag{12.45}$$

The magnetic induction of the potentials $A_0 = 0$ and $\boldsymbol{A} = B(x\hat{\boldsymbol{y}} - y\hat{\boldsymbol{x}})/2$ is $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$. □

In terms of the second-rank, antisymmetric Faraday field-strength tensor

$$F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = -F_{ji} \tag{12.46}$$

the electric field is $E_i = c\,F_{i0}$ and the magnetic field B_i is

$$B_i = \frac{1}{2}\epsilon_{ijk}\,F_{jk} = \frac{1}{2}\epsilon_{ijk}\left(\frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}\right) = (\nabla \times \boldsymbol{A})_i \tag{12.47}$$

where the sum over repeated indices runs from 1 to 3. The inverse equation $F_{jk} = \epsilon_{jki} B_i$ for spatial j and k follows from the Levi-Civita identity (1.535)

$$\begin{aligned}\epsilon_{jki} B_i &= \frac{1}{2}\epsilon_{jki}\epsilon_{inm}\,F_{nm} = \frac{1}{2}\epsilon_{ijk}\epsilon_{inm}\,F_{nm} \\ &= \frac{1}{2}\left(\delta_{jn}\,\delta_{km} - \delta_{jm}\,\delta_{kn}\right) F_{nm} = \frac{1}{2}\left(F_{jk} - F_{kj}\right) = F_{jk}.\end{aligned} \tag{12.48}$$

In 3-vector notation and MKSA = SI units, Maxwell's equations are a ban on magnetic monopoles and **Faraday's law**, both homogeneous,

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \tag{12.49}$$

and **Gauss's laws** and the **Maxwell-Ampère law**, both inhomogeneous,

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho_f \quad \text{and} \quad \boldsymbol{\nabla} \times \boldsymbol{H} = \boldsymbol{j}_f + \dot{\boldsymbol{D}}. \tag{12.50}$$

Here $\boldsymbol{H}$ is the **magnetic field**, ρ_f is the density of **free** charge and $\boldsymbol{j}_f$ is the **free current density**. By *free*, we understand charges and currents that do not arise from polarization and are not restrained by chemical bonds. The divergence of $\nabla \times \boldsymbol{H}$ vanishes (like that of any curl), and so the Maxwell-Ampère law and Gauss's law imply that free charge is conserved

$$0 = \nabla \cdot (\nabla \times \boldsymbol{H}) = \nabla \cdot \boldsymbol{j}_f + \nabla \cdot \dot{\boldsymbol{D}} = \nabla \cdot \boldsymbol{j}_f + \dot{\rho}_f. \tag{12.51}$$

If we use this **continuity equation** to replace $\nabla \cdot \boldsymbol{j}_f$ with $-\dot{\rho}_f$ in its middle form $0 = \nabla \cdot \boldsymbol{j}_f + \nabla \cdot \dot{\boldsymbol{D}}$, then we see that the Maxwell-Ampère law preserves the Gauss-law constraint in time

$$0 = \nabla \cdot \boldsymbol{j}_f + \nabla \cdot \dot{\boldsymbol{D}} = \frac{\partial}{\partial t}\left(-\rho_f + \nabla \cdot \boldsymbol{D}\right). \tag{12.52}$$

Similarly, Faraday's law preserves the constraint $\nabla \cdot \boldsymbol{B} = 0$

$$0 = -\nabla \cdot (\nabla \times \boldsymbol{E}) = \frac{\partial}{\partial t} \nabla \cdot \boldsymbol{B} = 0. \tag{12.53}$$

In a **linear, isotropic** medium, the **electric displacement** $\boldsymbol{D}$ is related to the electric field $\boldsymbol{E}$ by the **permittivity** ϵ, $\boldsymbol{D} = \epsilon \boldsymbol{E}$, and the magnetic or magnetizing field $\boldsymbol{H}$ differs from the magnetic induction $\boldsymbol{B}$ by the **permeability** μ, $\boldsymbol{H} = \boldsymbol{B}/\mu$.

On a sub-nanometer scale, the microscopic form of Maxwell's equations applies. On this scale, the homogeneous equations (12.49) are unchanged, but the inhomogeneous ones are

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla \times \boldsymbol{B} = \mu_0 \, \boldsymbol{j} + \epsilon_0 \, \mu_0 \, \dot{\boldsymbol{E}} = \mu_0 \, \boldsymbol{j} + \frac{\dot{\boldsymbol{E}}}{c^2} \tag{12.54}$$

in which ρ and $\boldsymbol{j}$ are the total charge and current densities, and $\epsilon_0 = 8.854 \times 10^{-12}$ F/m and $\mu_0 = 4\pi \times 10^{-7}$ N/A^2 are the **electric** and **magnetic constants**, whose product is the inverse of the square of the speed of light, $\epsilon_0 \mu_0 = 1/c^2$. Gauss's law and the Maxwell-Ampère law (12.54) imply (exercise 12.16) that the microscopic (total) current-density 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho} + \nabla \cdot \boldsymbol{j} = 0$. Electric charge is conserved.

In vacuum, $\rho = \boldsymbol{j} = 0$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, and $\boldsymbol{H} = \boldsymbol{B}/\mu_0$, and Maxwell's equations become

$$\begin{aligned} \nabla \cdot \boldsymbol{B} = 0 \quad &\text{and} \quad \nabla \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0 \\ \nabla \cdot \boldsymbol{E} = 0 \quad &\text{and} \quad \nabla \times \boldsymbol{B} = \frac{1}{c^2} \dot{\boldsymbol{E}}. \end{aligned} \tag{12.55}$$

Two of these equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{E} = 0$ are constraints. Taking the curl of the other two equations, we find

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{1}{c^2} \ddot{\boldsymbol{E}} \quad \text{and} \quad \nabla \times (\nabla \times \boldsymbol{B}) = -\frac{1}{c^2} \ddot{\boldsymbol{B}}. \tag{12.56}$$

One may use the Levi-Civita identity (1.535) to show (exercise 12.19) that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \triangle \boldsymbol{E} \text{ and } \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \triangle \boldsymbol{B} \tag{12.57}$$

in which $\triangle \equiv \boldsymbol{\nabla}^2$. Since in vacuum the divergence of $\boldsymbol{E}$ vanishes, and since that of $\boldsymbol{B}$ always vanishes, these identities and the curl-curl equations (12.56) tell us that waves of $\boldsymbol{E}$ and $\boldsymbol{B}$ move at the speed of light

$$\frac{1}{c^2}\ddot{\boldsymbol{E}} - \triangle \boldsymbol{E} = 0 \quad \text{and} \quad \frac{1}{c^2}\ddot{\boldsymbol{B}} - \triangle \boldsymbol{B} = 0. \tag{12.58}$$

We may write the two homogeneous Maxwell equations (12.49) as

$$\begin{aligned} \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = {} & \partial_i \left(\partial_j A_k - \partial_k A_j\right) + \partial_k \left(\partial_i A_j - \partial_j A_i\right) \\ & + \partial_j \left(\partial_k A_i - \partial_i A_k\right) = 0 \end{aligned} \tag{12.59}$$

(exercise 12.17). This relation, known as the **Bianchi identity**, actually is a generally covariant tensor equation

$$\epsilon^{\ell ijk} \partial_i F_{jk} = 0 \tag{12.60}$$

in which $\epsilon^{\ell ijk}$ is totally antisymmetric, as explained in Sec. 13.22. There are four versions of this identity (corresponding to the four ways of choosing three different indices i, j, k from among four and leaving out one, ℓ). The $\ell = 0$ case gives the scalar equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, and the three that have $\ell \neq 0$ give the vector equation $\boldsymbol{\nabla} \times \boldsymbol{E} + \dot{\boldsymbol{B}} = 0$.

In tensor notation, the microscopic form of the two inhomogeneous equations (12.54)—the laws of Gauss and Ampère—are a single equation

$$\partial_i F^{ki} = \mu_0 \, j^k \quad \text{in which} \quad j^k = (c\rho, \boldsymbol{j}) \tag{12.61}$$

is the current 4-vector.

The **Lorentz force law** for a particle of charge q is

$$m \, \frac{d^2 x^i}{d\tau^2} = m \, \frac{du^i}{d\tau} = \frac{dp^i}{d\tau} = f^i = q \, F^{ij} \, \frac{dx_j}{d\tau} = q \, F^{ij} \, u_j. \tag{12.62}$$

Multiplying this equation by $d\tau/dt = \sqrt{1 - \boldsymbol{v}^2/c^2}$, we find for $i = 1, 2, 3$

$$\frac{dp^i}{dt} = q \left(-F^{i0} + \epsilon_{ijk} B_k v_j\right) \quad \text{or} \quad \frac{d\boldsymbol{p}}{dt} = q \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \tag{12.63}$$

and for $i = 0$

$$\frac{dE}{dt} = q \, \boldsymbol{E} \cdot \boldsymbol{v} \tag{12.64}$$

which shows that only the electric field does work. The only special-relativistic correction needed in Maxwell's electrodynamics is a factor of $1/\sqrt{1 - v^2/c^2}$

in these equations. That is, we use $\boldsymbol{p} = m\boldsymbol{u} = m\boldsymbol{v}/\sqrt{1 - v^2/c^2}$ not $\boldsymbol{p} = m\boldsymbol{v}$ in (12.63), and we use the total energy E not the kinetic energy in (12.64). The reason why so little of classical electrodynamics was changed by special relativity is that electric and magnetic effects were accessible to measurement during the 1800's. Classical electrodynamics was almost perfect.

Keeping track of factors of the speed of light is a lot of trouble and a distraction; in what follows, we'll often use units with $c = 1$.

12.5 Principle of stationary action in special relativity

The action for a free particle of mass m in special relativity is

$$S = \, - m \int_{\tau_1}^{\tau_2} d\tau = \, - \int_{t_1}^{t_2} m\sqrt{1 - \dot{\boldsymbol{x}}^2}\, dt \tag{12.65}$$

where $c = 1$ and $\dot{\boldsymbol{x}} = d\boldsymbol{x}/dt$. The requirement of stationary action is

$$0 = \delta S = \, - \delta \int_{t_1}^{t_2} m\sqrt{1 - \dot{\boldsymbol{x}}^2}\, dt = m \int_{t_1}^{t_2} \frac{\dot{\boldsymbol{x}} \cdot \delta\dot{\boldsymbol{x}}}{\sqrt{1 - \dot{\boldsymbol{x}}^2}}\, dt. \tag{12.66}$$

But $1/\sqrt{1 - \dot{\boldsymbol{x}}^2} = dt/d\tau$ and so

$$\begin{aligned} 0 = \delta S &= m \int_{t_1}^{t_2} \frac{d\boldsymbol{x}}{dt} \cdot \frac{d\delta\boldsymbol{x}}{dt} \frac{dt}{d\tau}\, dt = m \int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{dt} \cdot \frac{d\delta\boldsymbol{x}}{dt} \frac{dt}{d\tau} \frac{dt}{d\tau}\, d\tau \\ &= m \int_{\tau_1}^{\tau_2} \frac{d\boldsymbol{x}}{d\tau} \cdot \frac{d\delta\boldsymbol{x}}{d\tau}\, d\tau. \end{aligned} \tag{12.67}$$

So integrating by parts, keeping in mind that $\delta\boldsymbol{x}(\tau_2) = \delta\boldsymbol{x}(\tau_1) = \boldsymbol{0}$, we have

$$0 = \delta S = m \int_{\tau_1}^{\tau_2} \left[\frac{d}{d\tau} \left(\frac{d\boldsymbol{x}}{d\tau} \cdot \delta\boldsymbol{x} \right) - \frac{d^2\boldsymbol{x}}{d\tau^2} \cdot \delta\boldsymbol{x} \right] d\tau = -m \int_{\tau_1}^{\tau_2} \frac{d^2\boldsymbol{x}}{d\tau^2} \cdot \delta\boldsymbol{x}\, d\tau. \tag{12.68}$$

To have this hold for arbitrary $\delta\boldsymbol{x}$, we need

$$\frac{d^2\boldsymbol{x}}{d\tau^2} = \boldsymbol{0} \tag{12.69}$$

which is the equation of motion for a free particle in special relativity.

What about a charged particle in an electromagnetic field A_i? Its action is

$$S = \, - m \int_{\tau_1}^{\tau_2} d\tau + q \int_{x_1}^{x_2} A_i(x)\, dx^i = \int_{\tau_1}^{\tau_2} \left(-m + qA_i(x) \frac{dx^i}{d\tau} \right) d\tau. \tag{12.70}$$

We now treat the first term in a four-dimensional manner

$$\delta d\tau = \delta\sqrt{-\eta_{ik}dx^i dx^k} = \frac{-\eta_{ik}dx^i\delta dx^k}{\sqrt{-\eta_{ik}dx^i dx^k}} = -u_k\delta dx^k = -u_k d\delta x^k \quad (12.71)$$

in which $u_k = dx_k/d\tau$ is the 4-velocity (12.31) and η is the Minkowski metric (12.3) of flat spacetime. The variation of the other term is

$$\delta\left(A_i\, dx^i\right) = \left(\delta A_i\right)\, dx^i + A_i\,\delta dx^i = A_{i,k}\delta x^k\, dx^i + A_i\, d\delta x^i. \quad (12.72)$$

Putting them together, we get for δS

$$\delta S = \int_{\tau_1}^{\tau_2}\left(m u_k\frac{d\delta x^k}{d\tau} + qA_{i,k}\delta x^k\frac{dx^i}{d\tau} + qA_i\frac{d\delta x^i}{d\tau}\right)d\tau. \quad (12.73)$$

After integrating by parts the last term, dropping the boundary terms, and changing a dummy index, we get

$$\begin{aligned}\delta S &= \int_{\tau_1}^{\tau_2}\left(-m\frac{du_k}{d\tau}\delta x^k + qA_{i,k}\delta x^k\frac{dx^i}{d\tau} - q\frac{dA_k}{d\tau}\delta x^k\right)d\tau \\ &= \int_{\tau_1}^{\tau_2}\left[-m\frac{du_k}{d\tau} + q\left(A_{i,k} - A_{k,i}\right)\frac{dx^i}{d\tau}\right]\delta x^k\, d\tau. \end{aligned} \quad (12.74)$$

If this first-order variation of the action is to vanish for arbitrary δx^k, then the particle must follow the path

$$0 = -m\frac{du_k}{d\tau} + q\left(A_{i,k} - A_{k,i}\right)\frac{dx^i}{d\tau} \qquad \text{or} \qquad \frac{dp^k}{d\tau} = qF^{ki}u_i \quad (12.75)$$

which is the Lorentz force law (12.62).

12.6 Differential forms

According to the chain rule of calculus, under arbitrary general coordinate transformations $x'^i = x'^i(x)$, coordinate differentials dx^i transform as

$$dx'^i = \frac{\partial x'^i}{\partial x^k}\, dx^k \quad (12.76)$$

and partial derivatrives transform as

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^k}{\partial x'^i}\frac{\partial}{\partial x^k}. \quad (12.77)$$

Vectors A_i that transform like partial derivatives are called **covariant** vectors, and ones B^k that transform like coordinate differentials are called **contravariant** vectors.

Again by the chain rule, a covariant vector contracted with contravariant coordinate differentials is invariant under general coordinate transformations

$$A' = A'_i \, dx'^i = \frac{\partial x^j}{\partial x'^i} \, A_j \, \frac{\partial x'^i}{\partial x^k} \, dx^k = \delta^j_k \, A_j \, dx^k = A_k \, dx^k = A. \tag{12.78}$$

This invariant quantity $A = A_k \, dx^k$ is a called a **1-form** in the language of **differential forms** introduced by Élie Cartan, son of a blacksmith.

The **wedge product** $dx \wedge dy$ of two coordinate differentials is the directed area spanned by the two differentials and is defined to be antisymmetric

$$dx \wedge dy = - \, dy \wedge dx \quad \text{and} \quad dx \wedge dx = dy \wedge dy = 0 \tag{12.79}$$

so as to transform correctly under a change of coordinates. In terms of the coordinates $u = u(x, y)$ and $v = v(x, y)$, the new element of area is

$$du \wedge dv = \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) \wedge \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right). \tag{12.80}$$

Labeling partial derivatives by subscripts (2.7) and using the antisymmetry (12.79) of the wedge product, we see that $du \wedge dv$ is the old area $dx \wedge dy$ multiplied by the Jacobian (section 1.23) of the transformation $x, y \to u, v$

$$\begin{aligned} du \wedge dv &= (u_x dx + u_y dy) \wedge (v_x dx + v_y dy) \\ &= u_x \, v_x \, dx \wedge dx + u_x \, v_y \, dx \wedge dy + u_y \, v_x \, dy \wedge dx + u_y \, v_y \, dy \wedge dy \\ &= (u_x \, v_y - u_y v_x) \; dx \wedge dy \\ &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} dx \wedge dy = J(u, v; x, y) \, dx \wedge dy. \end{aligned} \tag{12.81}$$

A contraction $H = \frac{1}{2} H_{ik} \, dx^i \wedge dx^k$ of a second-rank covariant tensor with a wedge product of two differentials is a 2-form. A ***p*-form** is a rank-p covariant tensor contracted with a wedge product of p differentials

$$K = \frac{1}{p!} \, K_{i_1 \dots i_p} \, dx^{i_1} \wedge \dots dx^{i_p}. \tag{12.82}$$

The **exterior derivative** d differentiates and adds a differential. It turns a p-form into a $(p+1)$-form. It turns a function f, which is a **0-form**, into a 1-form

$$df = \frac{\partial f}{\partial x^i} \, dx^i \tag{12.83}$$

and a 1-form $A = A_j \, dx^j$ into a 2-form $dA = d(A_j \, dx^j) = (\partial_i A_j) \, dx^i \wedge dx^j$.

Example 12.9 (The Curl) The exterior derivative of the 1-form

$$A = A_x\, dx + A_y\, dy + A_z\, dz \tag{12.84}$$

is a 2-form that contains the curl (2.44) of $\boldsymbol{A}$

$$\begin{aligned} dA &= \partial_y A_x\, dy \wedge dx + \partial_z A_x\, dz \wedge dx \\ &\quad + \partial_x A_y\, dx \wedge dy + \partial_z A_y\, dz \wedge dy \\ &\quad + \partial_x A_z\, dx \wedge dz + \partial_y A_z\, dy \wedge dz \\ &= (\partial_y A_z - \partial_z A_y)\, dy \wedge dz \\ &\quad + (\partial_z A_x - \partial_x A_z)\, dz \wedge dx \\ &\quad + (\partial_x A_y - \partial_y A_x)\, dx \wedge dy \\ &= (\nabla \times A)_x\, dy \wedge dz + (\nabla \times A)_y\, dz \wedge dx + (\nabla \times A)_z\, dx \wedge dy. \end{aligned} \tag{12.85}$$

□

The exterior derivative of the electromagnetic 1-form $A = A_j\, dx^j$ made from the 4-vector potential A_j is the Faraday 2-form (12.46), the tensor F_{ij}

$$dA = d\left(A_j\, dx^j\right) = \partial_i A_j\, dx^i \wedge dx^j = \tfrac{1}{2}\, F_{ij}\, dx^i \wedge dx^j = F \tag{12.86}$$

in which $\partial_i = \partial/\partial x^i$.

The square dd of the exterior derivative d vanishes when applied to any p-form Q

$$\begin{aligned} d\left[d\left(Q_{i\ldots} dx^i \wedge \ldots\right)\right] &= d\left[(\partial_r Q_{i\ldots})\, dx^r \wedge dx^i \wedge \ldots\right] \\ &= (\partial_s \partial_r Q_{i\ldots})\, dx^s \wedge dx^r \wedge dx^i \wedge \ldots = 0 \end{aligned} \tag{12.87}$$

because $\partial_s \partial_r Q$ is symmetric in r and s while $dx^s \wedge dx^r$ is anti-symmetric.

If M_{ik} is a covariant second-rank tensor with no particular symmetry, then (exercise 12.18) only its antisymmetric part contributes to the 2-form $M_{ik}\, dx^i \wedge dx^k$ and only its symmetric part contributes to $M_{ik}\, dx^i dx^k$.

Example 12.10 (The homogeneous Maxwell equations) The exterior derivative d applied to the Faraday 2-form $F = dA$ gives the homogeneous Maxwell equations

$$0 = ddA = dF = d\big(F_{ik}\, dx^i \wedge dx^k\big) = \partial_\ell F_{ik}\, dx^\ell \wedge dx^i \wedge dx^k \tag{12.88}$$

an equation known as the Bianchi identity (12.60). □

A p-form H is **closed** if $dH = 0$. By (12.88), the Faraday 2-form is closed,

$dF = 0$. A p-form H is **exact** if it is the differential $H = dK$ of a $(p-1)$-form K. The identity (12.87) or $dd = 0$ implies that **every exact form is closed**. A lemma (section 14.5) due to Poincaré shows that **every closed form is locally exact**.

If the A_i in the 1-form $A = A_i dx^i$ commute with each other, then the 2-form $A \wedge A$ is identically zero. But if the A_i don't commute because they are matrices, operators, or Grassmann variables, then $A \wedge A = \frac{1}{2}[A_i, A_j]\, dx^i \wedge dx^j$ need not vanish.

Example 12.11 (If $\dot{B} = 0$, the electric field is closed and exact) If $\dot{\boldsymbol{B}} = 0$, then by Faraday's law (12.49) the curl of the electric field vanishes, $\boldsymbol{\nabla} \times \boldsymbol{E} = 0$. In terms of the 1-form $E = E_i\, dx^i$ for $i = 1, 2, 3$, the vanishing of its curl $\boldsymbol{\nabla} \times \boldsymbol{E}$ is

$$dE = \partial_j E_i\, dx^j \wedge dx^i = \frac{1}{2}\left(\partial_j E_i - \partial_i E_j\right) dx^j \wedge dx^i = 0. \tag{12.89}$$

So E is closed. It also is exact because we can define a quantity $V(\boldsymbol{x})$ whose gradient is $\boldsymbol{E} = -\nabla V$. We first define $V_P(\boldsymbol{x})$ as a line integral of the 1-form E along an arbitrary path P from some starting point $\boldsymbol{x}_0$ to $\boldsymbol{x}$

$$V_P(\boldsymbol{x}) = -\int_{P,\, \boldsymbol{x}_0}^{\boldsymbol{x}} E_i\, dx^i = -\int_P E. \tag{12.90}$$

The potential $V_P(\boldsymbol{x})$ might seem to depend on the path P. But the difference $V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x})$ is a line integral of E from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ along the path P' and then back to $\boldsymbol{x}_0$ along the path P. And by Stokes's theorem (2.52), the integral of $\boldsymbol{E}$ around such a closed loop is an integral of the curl $\boldsymbol{\nabla} \times \boldsymbol{E}$ of $\boldsymbol{E}$ over any surface S whose boundary is that closed loop.

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \oint_{P-P'} E_i\, dx^i = \int_S (\boldsymbol{\nabla} \times \boldsymbol{E}) \cdot da = 0. \tag{12.91}$$

In the notation of forms, this is

$$V_{P'}(\boldsymbol{x}) - V_P(\boldsymbol{x}) = \int_{\partial S} E = \int_S dE = 0. \tag{12.92}$$

Thus the potential $V_P(\boldsymbol{x}) = V(\boldsymbol{x})$ is independent of the path, and $\boldsymbol{E} = -\boldsymbol{\nabla} V(\boldsymbol{x})$, and so the 1-form $E = E_i\, dx^i = -\partial_i V\, dx^i = -dV$ is exact. □

The general form of Stokes's theorem is that the integral of any p-form H over the boundary ∂R of any $(p+1)$-dimensional, simply connected, orientable region R is equal to the integral of the $(p+1)$-form dH over R

$$\int_{\partial R} H = \int_R dH. \tag{12.93}$$

Equation (12.92) is the $p = 1$ case (George Stokes, 1819–1903).

Example 12.12 (Stokes's theorem for 0-forms) When $p = 0$, the region $R = [a, b]$ is 1-dimensional, H is a 0-form, and Stokes's theorem is the formula of elementary calculus

$$H(b) - H(a) = \int_{\partial R} H = \int_R dH = \int_a^b dH(x) = \int_a^b H'(x)\, dx. \tag{12.94}$$

□

Example 12.13 (Exterior derivatives anticommute with differentials) The exterior derivative acting on the wedge product of two 1-forms $A = A_i dx^i$ and $B = B_\ell dx^\ell$ is

$$\begin{aligned} d(A \wedge B) &= d(A_i dx^i \wedge B_\ell dx^\ell) = \partial_k (A_i B_\ell)\, dx^k \wedge dx^i \wedge dx^\ell \qquad (12.95)\\ &= (\partial_k A_i)\, B_\ell\, dx^k \wedge dx^i \wedge dx^\ell + A_i\, (\partial_k B_\ell)\, dx^k \wedge dx^i \wedge dx^\ell \\ &= (\partial_k A_i)\, B_\ell\, dx^k \wedge dx^i \wedge dx^\ell - A_i\, (\partial_k B_\ell)\, dx^i \wedge dx^k \wedge dx^\ell \\ &= (\partial_k A_i)\, dx^k \wedge dx^i \wedge B_\ell dx^\ell - A_i dx^i \wedge (\partial_k B_\ell)\, dx^k \wedge dx^\ell \\ &= dA \wedge B - A \wedge dB. \end{aligned}$$

If A is a p-form, then $d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB$ (exercise 12.20). □

Special relativity is closely related general relativity. In fact, Einstein's **equivalence principle** says that special relativity applies in a suitably small neighborhood of any point in any inertial reference frame that has free-fall coordinates.

Exercises

12.1 (a) Are the events noon on Earth and noon on the Sun timelike, spacelike, or null? (b) A proton of energy 6.5 TeV in the LHC passes the same point $\boldsymbol{x}$ at times t_1 and t_2. Are the events, $(t_1, \boldsymbol{x})$ and $(t_2, \boldsymbol{x})$ timelike, spacelike, or null?

12.2 Show that (12.6) implies (12.7). For extra credit, show that the invariance of the quadratic distance (12.5) implies (12.7).

12.3 Show that the matrix form of the Lorentz transformation (12.17) is the x boost (12.18).

12.4 Show that the Lorentz matrix (12.18) satisfies ${L^i}_k\, \eta_{ij} {L^j}_\ell = \eta_{k\ell}$.

12.5 The basis vectors at a point p are the derivatives of the point with respect to the coordinates. Find the basis vectors $e_i = \partial p / \partial x^i$ at the point $p = (x^0, x^1, x^2, x^3)$. What are the basis vectors e'_i in the coordinates x' (12.19)?

12.6 The basis vectors e^i that are dual to the basis vectors e_k are defined by $e^i = \eta^{ik}\, e_k$. (a) Show that they obey $e^i \cdot e_k = \delta^i_k$. (b) In the two coordinate systems described in example 12.1, the vectors $e_k\, x^k$ and $e'_i\, x'^i$ represent the same point, so $e_k\, x^k = e'_i\, x'^i$. Find a formula for $e'^i \cdot e_k$. (c) Relate $e'^i \cdot e_k$ to the Lorentz matrix (12.18).

12.7 Show that the equality of the inner products $x^i \eta_{ik} x^k = x'^j \eta_{j\ell} x'^\ell$ means that the matrix $L^i{}_k = e'^i \cdot e_k$ that relates the coordinates $x'^i = L^i{}_k\, x^k$ to the coordinates x^k must obey the relation $\eta_{ik} = L^i{}_k\, \eta_{i\ell}\, L^\ell{}_k$. Hint: First doubly differentiate the equality with respect to x^k and to x^ℓ for $k \neq \ell$. Then differentiate it twice with respect to x^k.

12.8 The relations $x'^i = e'^i \cdot e_j\, x^j$ and $x^\ell = e^\ell \cdot e'_k\, x^k$ imply (for fixed basis vectors e and e') that

$$\frac{\partial x'^i}{\partial x^j} = e'^i \cdot e_j = e_j \cdot e'^i = \eta_{j\ell}\, \eta^{ik} e^\ell \cdot e'_k = \eta_{j\ell}\, \eta^{ik} \frac{\partial x^\ell}{\partial x'^k}.$$

Use this equation to show that if A^i transforms (13.8) as a contravariant vector

$$A'^i = \frac{\partial x'^i}{\partial x^j}\, A^j, \tag{12.96}$$

then $A_\ell = \eta_{\ell j}\, A^j$ transforms covariantly (13.11)

$$A'_s = \frac{\partial x^\ell}{\partial x'^s} A_\ell.$$

The metric η also turns a covariant vector A_ℓ into its contravariant form $\eta^{k\ell} A_\ell = \eta^{k\ell} \eta_{\ell j} A^j = \delta^k_j A^j = A^k$.

12.9 Two photons collide head on and make a proton and an antiproton. What was the minimum energy that each photon had to have? The mass of a proton is 938.272081 MeV.

12.10 What would be the lifetime of a negative pion π^- moving in a cyclotron at speed $v/c = 0.9$? At rest its lifetime is $\tau = 2.6033 \times 10^{-8}$ s.

12.11 What is the 4-momentum p^i at time t of a particle of mass m whose position at time t is $\boldsymbol{x}_0 \exp(t/t_0)$?

12.12 The LHC is designed to collide 7 TeV protons against 7 TeV protons for a total collision energy of 14 TeV. Suppose one used a linear accelerator to fire a beam of protons at a target of protons at rest at one end of the accelerator. What energy would you need to see the same physics as at the LHC?

12.13 What is the minimum energy that a beam of pions must have to produce a sigma hyperon and a kaon by striking a proton at rest? The relevant masses (in MeV) are $m_{\Sigma^+} = 1189.4$, $m_{K^+} = 493.7$, $m_p = 938.3$, and $m_{\pi^+} = 139.6$.

12.14 Suppose a particle of 4-momentum p and mass $m > 0$ so that $p^2 = \boldsymbol{p}^2 - (p^0)^2 = -c^2 m^2$ absorbs a photon. Can the final state be the same particle with the same mass m but with a different 4-momentum q such that $q^2 = \boldsymbol{q}^2 - (q^0)^2 = -c^2 m^2$? If not, why not?

12.15 Show that the electric field $\boldsymbol{E}$ (12.43) and the magnetic induction $\boldsymbol{B}$ (12.42) are unchanged by the gauge transformation (12.44).

12.16 Use Gauss's law and the Maxwell-Ampère law (12.54) to show that the microscopic (total) current-density 4-vector $j = (c\rho, \boldsymbol{j})$ obeys the continuity equation $\dot{\rho} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$.

12.17 Derive the Bianchi identity (12.59) from the definition (12.46) of the Faraday field-strength tensor, and show that it implies the two homogeneous Maxwell equations (12.49).

12.18 Show that if M_{ik} is a covariant second-rank tensor with no particular symmetry, then only its antisymmetric part contributes to the 2-form $M_{ik}\, dx^i \wedge dx^k$ and only its symmetric part contributes to the quantity $M_{ik}\, dx^i dx^k$.

12.19 In rectangular coordinates, use the Levi-Civita identity (1.535) to derive the curl-curl equations (12.57).

12.20 Show that if A is a p-form, then $d(AB) = dA \wedge B + (-1)^p A \wedge dB$.

12.21 Show that if $\omega = a_{ij} dx^i \wedge dx^j / 2$ with $a_{ij} = -a_{ji}$, then

$$d\omega = \frac{1}{3!} \left(\partial_k a_{ij} + \partial_i a_{jk} + \partial_j a_{ki} \right) dx^i \wedge dx^j \wedge dx^k. \tag{12.97}$$