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Quantum Mechanics

Quantum mechanics applies many of the ideas of linear algebra (chapter 1) and vector calculus (chapter 2). These ideas can be illustrated by some examples.

24.1 Measured states as vectors or kets

A particle measured to be at the point $\boldsymbol{x}$ is represented by a vector or ket $|\boldsymbol{x}\rangle$, which is the tensor product of three kets $|\boldsymbol{x}\rangle = |x\rangle|y\rangle|z\rangle$. It is said to be in the **state** $|\boldsymbol{x}\rangle$. A particle measured to have momentum $\boldsymbol{p}$ is represented by a tensor-product vector or ket $|\boldsymbol{p}\rangle = |p_x\rangle|p_y\rangle|p_z\rangle$. It is said to be in the **state** $|\boldsymbol{p}\rangle$. Similarly, an electron measured to have energy E, squared angular momentum $\hbar^2\ell(\ell+1)$, and angular momentum $\hbar m$ in the z direction is said to be in the state $|E,\ell,m\rangle$ and is represented by tensor-product ket $|E,\ell,m\rangle = |E\rangle|\ell\rangle|m\rangle$. More generally, a particle measured to have the physical properties associated with a given state ψ is said to be in the state $|\psi\rangle$ and is represented by a vector or ket $|\psi\rangle$.

24.2 Amplitudes and probabilities as inner products

The (in general complex) amplitude for a particle in the state $|\psi\rangle$ to be measured to be at the point $\boldsymbol{x}$ is the inner product $\langle\boldsymbol{x}|\psi\rangle$ of the ket $|\psi\rangle$ with the bra $\langle\boldsymbol{x}|$. And the **probability** that a particle in the state $|\psi\rangle$ is measured to be at the point $\boldsymbol{x}$ is the absolute-value squared of the inner product $\langle\boldsymbol{x}|\psi\rangle$ of the ket $|\psi\rangle$ with the bra $\langle\boldsymbol{x}|$

$$P(x,\psi) = |\langle\boldsymbol{x}|\psi\rangle|^2 = |\psi(\boldsymbol{x})|^2. \tag{24.1}$$

Thus the probability that an electron in the state $|E,\ell,m\rangle$ is found to be at position $\boldsymbol{x}$ is $|\langle\boldsymbol{x}|E,\ell,m\rangle|^2$.

24.3 Physical observables as matrices or linear operators

Physical observables are represented by real diagonal (and therefore hermitian) matrices with their possible values arranged on their main diagonals. For example, the spin of the electron in the z direction is represented as the matrix

$$\sigma_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{24.2}$$

Similarly, the energy of a nonrelativistic electron near a proton is represented by a hamiltonian operator H whose diagonal form is an infinite-dimensional matrix with diagonal elements $E_n = -\alpha^2 mc^2/(2n^2)$ for $n = 1, 2, \ldots$ with $\alpha \approx 1/137$. The position $\boldsymbol{x}$ of a particle is represented by three position operators x, y, and z whose diagonal forms are infinite-dimensional matrices with the points of the real axis arranged consecutively along their main diagonals. The momentum $\boldsymbol{p}$ of a particle can be represented similarly.

The foregoing definitions make perfect sense in classical physics. What distinguishes quantum mechanics from classical physics is that the hermitian operators momentum $\boldsymbol{p}$ and energy H generate unitary transformations among the states or kets.

24.4 Momentum, energy, and unitary transformations

Heisenberg told us that the commutator of the operators x and p that represent the position and momentum of a particle in one dimension is

$$[x, p] \equiv x\,p - p\,x = i\hbar. \tag{24.3}$$

Using x' and p' for the eigenvalues of x and p, we see that the $\langle x'|$, $|p'\rangle$ matrix element of this equation is

$$\langle x'|[x,p]|p'\rangle = x'p'\langle x'|p'\rangle - \langle x'|p\,x|p'\rangle = i\hbar\langle x'|p'\rangle \tag{24.4}$$

which implies that

$$\langle x'|\,p = \frac{\hbar}{i}\,\frac{d}{dx'}\langle x'|. \tag{24.5}$$

Using this equation to form the inner product

$$\langle x'|\,p|p'\rangle = p'\,\langle x'|p'\rangle = \frac{\hbar}{i}\,\frac{d}{dx'}\langle x'|p'\rangle \tag{24.6}$$

and integrating, we find the explicit form of the unitary operator $\langle x'|p'\rangle$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\,e^{ix'p'/\hbar}. \tag{24.7}$$

Again using the differential form (24.4) of Heisenberg's equation (24.3), we see that the unitary operator $e^{-iap/\hbar}$ translates a state $|\psi\rangle$ by a

$$\langle x' - a|\psi\rangle = \exp\left(-a \frac{d}{dx'}\right) \langle x'|\psi\rangle = \langle x'| \exp\left(-\frac{ia}{\hbar}\, p\right) |\psi\rangle. \tag{24.8}$$

Another way to see this is to expand a state $|\psi\rangle$ in terms of the momentum eigenstates $|p'\rangle$

$$\psi(x) = \langle x|\psi\rangle = \int dp' \langle x|p'\rangle\langle p'|\psi\rangle = \int \frac{dp'}{\sqrt{2\pi\hbar}}\, e^{ixp'/\hbar}\langle p'|\psi\rangle \tag{24.9}$$

and to compare it with the translated state

$$\begin{aligned}\psi_a(x) = \langle x|e^{-iap/\hbar}|\psi\rangle &= \int dp' \langle x|e^{-iap/\hbar}|p'\rangle\langle p'|\psi\rangle \\ &= \int \frac{dp'}{\sqrt{2\pi\hbar}}\, e^{i(x-a)p'/\hbar}\langle p'|\psi\rangle = \psi(x-a).\end{aligned} \tag{24.10}$$

An observable represented by an operator $A(x)$ is translated in space by the unitary operator $\exp(-i\boldsymbol{a}\cdot\boldsymbol{p}/\hbar)$

$$A(\boldsymbol{x} + \boldsymbol{a}) = e^{i\boldsymbol{a}\cdot\boldsymbol{p}/\hbar} A(\boldsymbol{x}) e^{-i\boldsymbol{a}\cdot\boldsymbol{p}/\hbar} \tag{24.11}$$

and in time by the unitary operator $\exp(-it'H/\hbar)$

$$A(t + t') = e^{it'H/\hbar} A(t) e^{-it'H/\hbar}. \tag{24.12}$$

The $|x'\rangle$'s and $|p'\rangle$'s are two bases for the Hilbert space of one-dimensional quantum mechanics. They are orthogonal but have delta-function (section 2.6) normalization

$$\langle x'|x''\rangle = \delta(x' - x'') \quad \text{and} \quad \langle p'|p''\rangle = \delta(p' - p''). \tag{24.13}$$

The inner product $\langle x'|p'\rangle$ (24.7) of these two orthogonal bases with delta-function normalization is a unitary matrix (1.191)

$$\int \frac{dp'}{\sqrt{2\pi\hbar}} \langle x'|p'\rangle\langle p'|x''\rangle = \int \frac{dp'}{\sqrt{2\pi\hbar}}\, e^{i(x'-x'')p/\hbar} = \delta(x' - x'') \tag{24.14}$$

in which the Fourier representation (section 3.11) of the delta function (section 2.6) is used.