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SI Units and Natural Units

23.1 Standard International (SI) Units

In Standard International (SI) units, the definitions, values, and dimensions of some basic physical quantities are

$$\begin{aligned}c &= 2.997\,924\,58 \times 10^8 \text{ m s}^{-1} \text{ (exactly)} \\h &= 6.626\,070\,15 \times 10^{-34} \text{ J s (exactly)} \\ \hbar &\equiv \frac{h}{2\pi} = 1.054\,571\,817 \dots \times 10^{-34} \text{ J s} \\e &= 1.602\,176\,634 \times 10^{-19} \text{ C (exactly)} \\k_{\text{B}} &= 1.380\,649 \times 10^{-23} \text{ J K}^{-1} \text{ (exactly)} \\N_{\text{A}} &= 6.022\,140\,76 \times 10^{23} \text{ mol}^{-1} \text{ (exactly)} \\ \Delta\nu_{\text{Cs}} &= 9.192\,631\,770 \times 10^9 \text{ Hz (exactly)} \\G &= 6.67430(15) \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \\ \mu_0 &= (4\pi) \times 1.000\,000\,000\,55(15) \times 10^7 \text{ N A}^{-2} \\ \epsilon_0 &\equiv 1/(\mu_0 c^2) = 8.854\,187\,8128(13) \times 10^{-12} \text{ F m}^{-1} \\ \alpha &\equiv \frac{e^2}{4\pi\epsilon_0\hbar c} = \frac{1}{137.035\,999\,084(21)} = 7.297\,352\,5693(11) \times 10^{-3}\end{aligned} \tag{23.1}$$

in which $\text{N} = \text{J m}^{-1}$, $\text{Hz} = \text{s}^{-1}$, $\text{A} = \text{C s}^{-1}$, $\text{F} = \text{kg}^{-1} \text{ m}^{-2} \text{ s}^2 \text{ C}^2$, and $\Delta\nu_{\text{Cs}}$ is a hyperfine transition frequency of Cesium-133.

The frequency $\Delta\nu_{\text{Cs}}$ defines the Hz and therefore the second. The speed of light c defines the meter m in terms of the second s. Planck's constant h defines the Joule J in terms of the second s, which gives us the kilogram kg since we already know the meter and the second. Boltzmann's constant k_{B} defines the degree Kelvin K in terms of the Joule J. Finally the absolute

value e of the charge of the electron defines the Coulomb C. To remember that the fine-structure constant is $\alpha \sim .007$, think of James Bond.

23.2 Natural Units

The symbols \hbar , c , G , k_B , m_e and so forth clutter our formulas. To avoid such distractions, Planck and Hartree invented systems of **natural units** that replace such constants by unity. Planck's system is used in cosmology and in nuclear and particle physics; Hartree's is used in atomic and molecular physics.

23.3 Planck Units

Planck set his constant \hbar , the speed of light c , Newton's constant G , and Boltzmann's constant k_B equal to unity. In his units, $c = 1$, $G = 1$, $k_B = 1$, and $\hbar \equiv h/2\pi = 1$. The Planck length $L_P = \sqrt{\hbar G/c^3} \sim 1.6 \times 10^{-35}$ m, mass $M_P = \sqrt{\hbar c/G} \sim 2.1 \times 10^{-8}$ kg, time $T_P = \sqrt{\hbar G/c^5} \sim 5.4 \times 10^{-44}$ s, and temperature $T_{P, \text{temp}} = \sqrt{\hbar c^5/Gk_B^2} \sim 1.4 \times 10^{32}$ K are all unity in Planck units.

In a calculation done in Planck's units, a quantity represented as a mass m can have the dimensions of inverse time $= 1/T$, inverse length $= 1/L$, mass $= M$, energy E , temperature K , inverse time $= 1/T$, inverse length $= 1/L$, or inverse temperature K . A quantity represented as a mass m can be a momentum or an energy

$$[mc] = \frac{ML}{T} \quad \text{or} \quad [mc^2] = \frac{ML^2}{T^2} \quad (23.2)$$

or an inverse length or an inverse time

$$\left[\frac{mc}{\hbar} \right] = \frac{MLT}{TML^2} = \frac{1}{L} \quad \text{or} \quad \left[\frac{mc^2}{\hbar} \right] = \frac{E}{ET} = \frac{1}{T} \quad (23.3)$$

or a length or a time

$$\left[\frac{Gm}{c^2} \right] = \frac{L^3}{MT^2} \frac{MT^2}{L^2} = L \quad \text{or} \quad \left[\frac{Gm}{c^3} \right] = \frac{L^3}{MT^2} \frac{MT^3}{L^3} = T. \quad (23.4)$$

or a temperature or an inverse temperature

$$\left[\frac{m c^2}{k_B} \right] = \frac{EK}{E} = K \quad \text{or} \quad \left[\frac{m k_B G}{c^3 \hbar} \right] = \frac{ME}{K} \frac{L^3}{MT^2} \frac{T^3}{L^3 ET} = \frac{1}{K}. \quad (23.5)$$

Example 23.1 (Mass as inverse length, energy, and length) The range of the nuclear force is given approximately in terms of the mass of the pion

as $r \approx 1/m_\pi = \hbar c/m_\pi = 197 \text{ MeV fm}/139 \text{ MeV} = 1.4 \text{ fm} = 1.4 \times 10^{-15} \text{ m}$. The energy levels of a hydrogen atom are $E_n = -\frac{1}{2}\alpha^2 m_e/n^2 = -13.6/n^2 \text{ eV}$. The Schwarzschild radius of a point mass m as massive as the Earth is $r_s = 2m_E = 2Gm_E/c^2 = 2(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}) \times 5.97 \times 10^{24} \text{ kg}/(3 \times 10^8 \text{ m/s})^2 = 8.87 \text{ mm}$. \square

23.4 Hartree Units

Hartree set Planck's constant \hbar , the mass of the electron m_e , the charge of the proton e , and Coulomb's constant $k_e = 1/(4\pi\epsilon_0)$ equal to unity. The Hartree length $L_H = 4\pi\epsilon_0 \hbar^2/m_e e^2 \sim 5.3 \times 10^{-11} \text{ m}$, mass $M_H = m_e \sim 9.1 \times 10^{-31} \text{ kg}$, time $T_H = (4\pi\epsilon_0)^2 \hbar^2/m_e e^4 \sim 2.4 \times 10^{-17} \text{ s}$, and charge $Q_H = e \sim 1.6 \times 10^{-19} \text{ C}$ are all unity in Hartree units. In his units, the Bohr radius $a_0 = L_H = 1$, and the energy levels of a hydrogen atom are $E_n = 1/(2n^2)$.