gauge condition $h^i{}_{k,i} = \frac{1}{2}(\eta^{j\ell}h_{j\ell})_{,k} \equiv \frac{1}{2}h_{,k}$. In this gauge, the linearized Einstein equations in empty space are

$$R_{i\ell} = -\frac{1}{2}\,h_{i\ell,k}{}^{,k} = 0 \quad \text{or} \quad \left(c^2\nabla^2 - \partial_0^2\right)h_{i\ell} = 0. \tag{13.256}$$

In 2015, the LIGO collaboration detected the merger of two black holes of 29 and 36 solar masses which liberated $3M_{\odot}c^2$ of energy. They have set an upper limit of $c^2 m_g < 2 \times 10^{-25}$ eV on the mass of the graviton, have detected 10 black-hole mergers, and are expected to detect a new merger every week in late 2019.

13.39 Schwarzschild and Eddington Metrics

In 1916, Schwarzschild solved Einstein's field equations (13.251) in empty space $R_{ij} = 0$ outside a static, spherically symmetric mass M. He found

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\,d\Omega^2, \tag{13.257}$$

in which $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2$. The Mathematica scripts GREAT.m and Schwarzschild.nb show that the Schwarzschild metric obeys Einstein's equations (13.237) for empty space $R_{ik} = 0$ and $R = 0$.

Eddington set $r = [1 + GM/(2c^2 r')]^2 r'$ and got

$$\begin{aligned} ds^2 &= -\frac{\left(1 - \frac{GM}{2c^2 r'}\right)^2}{\left(1 + \frac{GM_e}{2c^2 r'}\right)^2}\,c^2 dt^2 + \left(1 + \frac{GM}{2c^2 r'}\right)^4 (dr'^2 + r'^2 d\Omega^2) \\ &= -\frac{\left(1 - \frac{GM}{2c^2 r'}\right)^2}{\left(1 + \frac{GM_e}{2c^2 r'}\right)^2}\,c^2 dt^2 + \left(1 + \frac{GM}{2c^2 r'}\right)^4 (dx^2 + dy^2 + dz^2) \end{aligned} \tag{13.258}$$

in which $r' = \sqrt{x^2 + y^2 + z^2}$ (Eddington, 1924). His metric is free of the unphysical singularity $g_{rr} = (1 - 2GM/c^2 r)^{-1}$ in Schwarzschild's metric (13.257) and is isotropic with the speed of light the same in all directions.

The proper time $d\tau$ measured by a clock hovering at r in Schwarzschild's coordinates is

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r}}\,dt \tag{13.259}$$

while that of a clock far from the mass M is $d\tau = dt$. At the Schwarzschild radius $r_{\text{S}} = 2GM/c^2$ in Schwarzschild's coordinates (and equivalently at the Eddington radius $r_{\text{E}} = \frac{1}{2}GM/c^2$ in Eddington's coordinates) the clock stops.

Light emitted by an atom at $r_{\rm S}$ or $r_{\rm E}$ is red-shifted to zero frequency. (Karl Schwarzschild 1873–1916, Arthur Eddington 1882–1944)

13.40 Black holes

Suppose an uncharged, static, spherically symmetric star of mass M has collapsed to within a sphere of radius less than $r_{\rm S} = 2MG/c^2$ in Schwarzschild's coordinates or equivalently less than $r_{\rm E} = \frac{1}{2}MG/c^2$ in Eddington's. Then outside the star, the metrics (13.257 and 13.258) are correct, and light emitted by the star is red-shifted to zero frequency. The star is a black hole. If the radius of the Sun, 6.957×10^5 km, were less than its Schwarzschild radius of 2.95 km, the Sun would be a black hole.

Black holes are not black. They often are surrounded by bright hot accretion disks, and Stephen Hawking showed (Hawking, 1975) that the intense gravitational field of a black hole of mass M radiates at a temperature

$$T = \frac{\hbar\, c^3}{8\pi\, k_{\rm B}\, G\, M} \tag{13.260}$$

in which $k_{\rm B} = 8.617 \times 10^{-5}\,\text{eV K}^{-1}$ is Boltzmann's constant, and $\hbar = 1.055 \times 10^{-34}\,\text{J s}$.

In a region of empty space where the pressure p and the chemical potentials μ_j vanish, the change (7.117) in the internal energy $U = c^2 M$ of a black hole of mass M is $dU = c^2 dM = T dS$ where S is its entropy. So the change dS in the entropy of a black hole of mass M and temperature T (13.260) is

$$dS = \frac{c^2}{T} dM = \frac{8\pi k_{\rm B} M dM}{\hbar c}. \tag{13.261}$$

Integrating, we get a formula for the entropy of a black hole in terms of its mass and also in terms of the areas of its Schwarzschild $A_{\rm S} = 4\pi r_{\rm S}^2$ and Eddington $A_{\rm E} = 4\pi r_{\rm E}^2$ horizons (Bekenstein, 1973)

$$S = \frac{4\pi k_{\rm B} G M^2}{\hbar c} = \frac{c^3 k_{\rm B}}{4\hbar G}\, A_{\rm S} = \frac{4c^3 k_{\rm B}}{\hbar G}\, A_{\rm E}. \tag{13.262}$$

The entropy of a black hole of 60 solar masses is about 4×10^{57}.

A black hole radiates energy according to the Stefan-Boltzmann law (5.110)

$$c^2 \frac{dM}{dt} = \sigma A T^4 = \sigma 4\pi r_{\rm S}^2 T^4 = \frac{\hbar c^6}{15360 \pi G^2 M^2}. \tag{13.263}$$

Integrating, we see that a black hole is entirely converted into radiation after a time

$$t = \frac{5120\,\pi\,G^2}{\hbar\,c^4}\,M^3 \tag{13.264}$$

proportional to the cube of its mass M. (Stephen Hawking 1942–2018)

13.41 Rotating black holes

A half-century after Einstein invented general relativity, Roy Kerr found the metric for a mass M rotating with angular momentum $J = cMa$. Two years later, Newman and others generalized the Kerr metric to one of charge q. In Boyer-Lindquist coordinates, its line element is

$$\begin{aligned} ds^2 = & -\frac{\Delta}{\rho^2}\left(c\,dt - a\,\sin^2\theta d\phi\right)^2 \\ & + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)d\phi - a\,c\,dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \\ = & -\left(1 - \frac{2GMr/c^2 - Q^2}{\rho^2}\right)c^2dt^2 - \frac{2a\,\sin^2\theta\,(2GMr/c^2 - Q^2)}{\rho^2}\,c\,dt\,d\phi \\ & + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\,\sin^2\theta\,d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{13.265}$$

in which $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2GMr/c^2 + a^2 + Q^2$. Here $Q^2 = G\,q^2/(4\pi\epsilon_0 c^4)$ and q is the charge in Coulombs. The Mathematica script Kerr_black_hole.nb shows that the Kerr-Newman metric for the uncharged case, $q = 0$, has $R_{ik} = 0$ and $R = 0$, and so is a solution of Einstein's equations in empty space (13.241) with zero scalar curvature.

A rotating mass drags nearby masses along with it. The daily rotation of the Earth drags satellites to the East by tens of meters per year. The **frame dragging** of extremal black holes can approach the speed of light. (Roy Kerr 1934–, Ezra Newman 1929–2021)

13.42 Friedman-Lemaître-Robinson-Walker Cosmologies

Einstein's equations (13.241) are second-order, nonlinear partial differential equations for 10 unknown functions $g_{ik}(x)$ in terms of the energy-momentum tensor $T_{ik}(x)$ throughout the universe, which of course we don't know. The