$\exp(-st)$ lets many functions $F(t)$ that are not integrable over the half line $[0,\infty)$ have well-behaved Laplace transforms.

For instance, the function $F(t) = 1$ is not integrable over the half line $[0,\infty)$, but its Laplace transform

$$f(s) = \int_0^\infty dt\, e^{-st}\, F(t) = \int_0^\infty dt\, e^{-st} = \frac{1}{s} \tag{4.130}$$

is well defined for $\mathrm{Re}\, s > 0$ and square integrable for $\mathrm{Re}\, s > \epsilon$.

The function $F(t) = \exp(kt)$ diverges exponentially for $\mathrm{Re}\, k > 0$, but its Laplace transform

$$f(s) = \int_0^\infty dt\, e^{-st}\, F(t) = \int_0^\infty dt\, e^{-(s-k)t} = \frac{1}{s-k} \tag{4.131}$$

is well defined for $\mathrm{Re}\, s > k$ with a simple pole at $s = k$ (section 6.10) and is square integrable for $\mathrm{Re}\, s > k + \epsilon$.

The Laplace transforms of $\cosh kt$ and $\sinh kt$ are

$$f(s) = \int_0^\infty dt\, e^{-st}\, \cosh kt = \frac{1}{2}\int_0^\infty dt\, e^{-st}\, \left(e^{kt} + e^{-kt}\right) = \frac{s}{s^2 - k^2} \tag{4.132}$$

and

$$f(s) = \int_0^\infty dt\, e^{-st}\, \sinh kt = \frac{1}{2}\int_0^\infty dt\, e^{-st}\, \left(e^{kt} - e^{-kt}\right) = \frac{k}{s^2 - k^2}. \tag{4.133}$$

The Laplace transform of $\cos \omega t$ is

$$f(s) = \int_0^\infty dt\, e^{-st}\, \cos \omega t = \frac{1}{2}\int_0^\infty dt\, e^{-st}\, \left(e^{i\omega t} + e^{-i\omega t}\right) = \frac{s}{s^2 + \omega^2} \tag{4.134}$$

and that of $\sin \omega t$ is

$$f(s) = \int_0^\infty dt\, e^{-st}\, \sin \omega t = \frac{1}{2i}\int_0^\infty dt\, e^{-st}\, \left(e^{i\omega t} - e^{-i\omega t}\right) = \frac{\omega}{s^2 + \omega^2}. \tag{4.135}$$

Example 4.13 (Lifetime of a Fluorophore) Fluorophores are molecules that emit visible light when excited by photons. The probability $P(t,t')$ that a fluorophore with a lifetime τ will emit a photon at time t if excited by a photon at time t' is

$$P(t,t') = \tau^{-1}\, e^{-(t-t')/\tau}\, \theta(t-t') \tag{4.136}$$

in which $\theta(t-t') = (t - t' + |t-t'|)/2|t-t'|$ is the Heaviside function. One way to measure the lifetime τ of a fluorophore is to modulate the exciting laser beam at a frequency $\nu = 2\pi\omega$ of the order of 60 MHz and to detect the phase-shift ϕ in the light $L(t)$ emitted by the fluorophore. That light is the integral of $P(t,t')$ times the modulated beam $\sin \omega t$ or equivalently the convolution of $e^{-t/\tau}\theta(t)$ with $\sin \omega t$

$$L(t) = \int_{-\infty}^{\infty} P(t,t')\, \sin(\omega t')\, dt' = \int_{-\infty}^{\infty} \tau^{-1}\, e^{-(t-t')/\tau} \theta(t-t')\, \sin(\omega t')\, dt'$$
$$= \int_{-\infty}^{t} \tau^{-1}\, e^{-(t-t')/\tau} \sin(\omega t')\, dt'. \quad (4.137)$$

Letting $u = t - t'$ and using the trigonometric formula

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \quad (4.138)$$

we may relate this integral to the Laplace transforms of a sine (4.135) and a cosine (4.134)

$$L(t) = -\tau^{-1} \int_0^{\infty} e^{-u/\tau}\, \sin \omega(u-t)\, du$$
$$= -\tau^{-1} \int_0^{\infty} e^{-u/\tau}\, (\sin \omega u \cos \omega t - \cos \omega u \sin \omega t)\, du$$
$$= \frac{\sin(\omega t) - \omega\tau \cos(\omega t)}{1 + (\omega\tau)^2}. \quad (4.139)$$

Setting $\cos\phi = 1/\sqrt{1+(\omega\tau)^2}$ and $\sin\phi = \omega\tau/\sqrt{1+(\omega\tau)^2}$, we have

$$L(t) = \frac{1}{\sqrt{1+(\omega\tau)^2}}\, (\sin \omega t \cos\phi - \cos \omega t \sin\phi) = \frac{1}{\sqrt{1+(\omega\tau)^2}}\, \sin(\omega t - \phi). \quad (4.140)$$

The phase-shift ϕ then is given by

$$\phi = \arctan(\omega\tau) \le \frac{\pi}{2} \quad (4.141)$$

and the lifetime of the fluorophore in terms of the phase-shift ϕ

$$\tau = (1/\omega)\, \tan\phi \quad (4.142)$$

which is much easier to measure than the lifetime τ. □

4.10 Inversion of Laplace transforms

How do we invert the Laplace transform

$$f(s) = \int_0^{\infty} dt\, e^{-st}\, F(t)? \quad (4.143)$$

First we extend the Laplace transform from real s to $s + iu$

$$f(s+iu) = \int_0^{\infty} dt\, e^{-(s+iu)t}\, F(t) \quad (4.144)$$