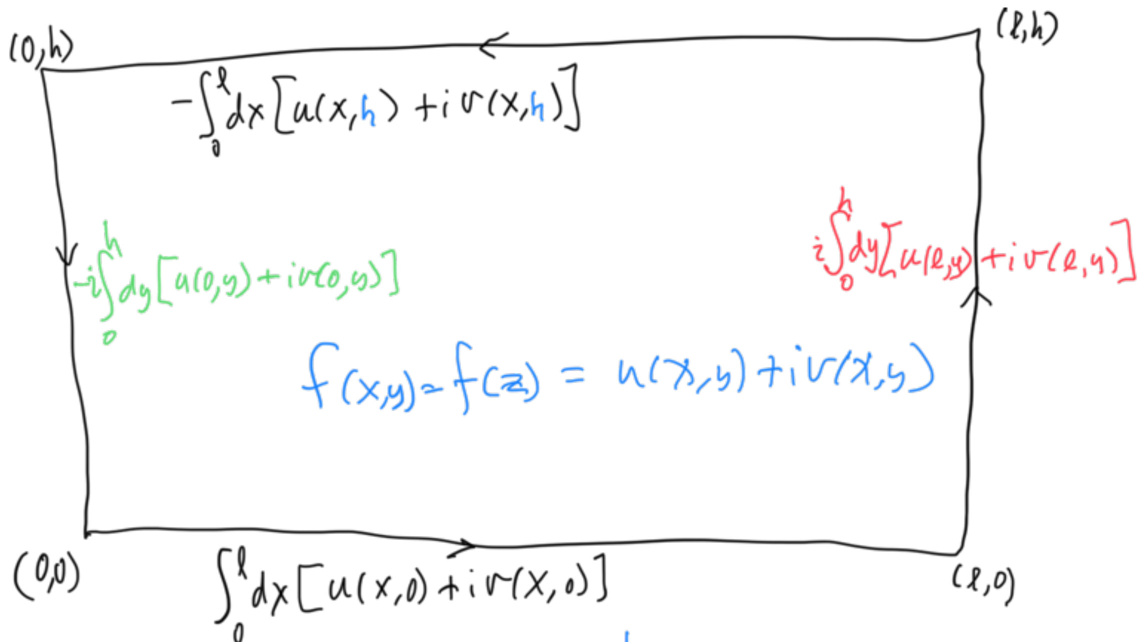


$$\mathrm{Re}\left(\oint f(z)\,dz\right)=\int_0^l dx\,[u(x,0)-u(x,h)]-\int_0^h dy\,[v(l,y)-v(0,y)]$$

$f(z)=u(x,y)+iv(x,y)$

$$u(x,0)-u(x,h)=-\int_0^h dy\,u_y(x,y) \qquad v(l,y)-v(0,y)=\int_0^l dx\,v_x(x,y)$$

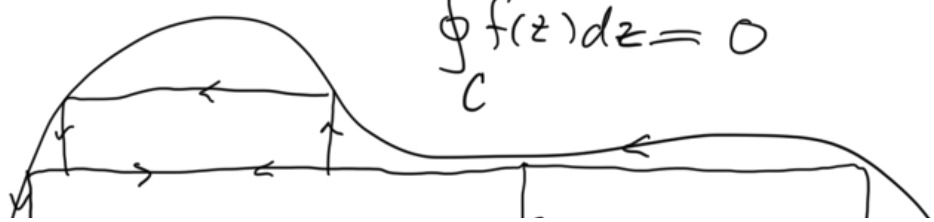
$$=-\int_0^l dx\int_0^h dy\,u_y(x,y)-\int_0^l dx\int_0^h dy\,v_x(x,y)=0 \quad \text{since} \quad v_x=-u_y.$$

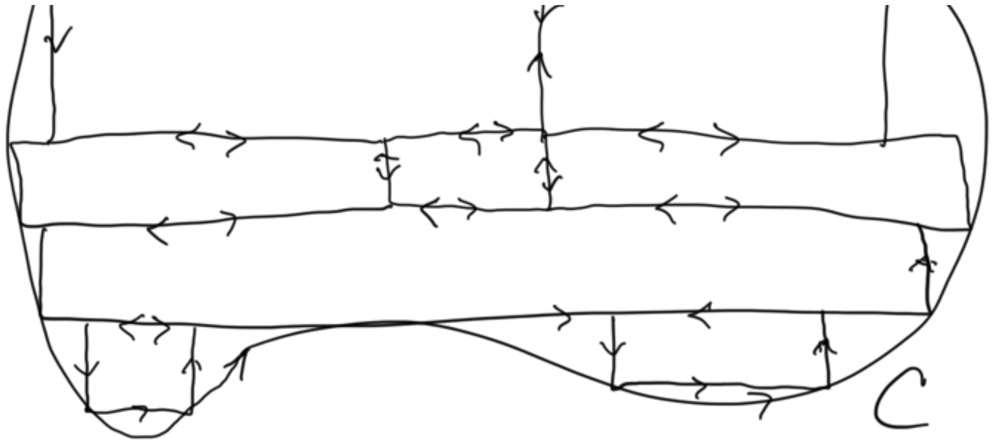
$$\mathrm{Im}\left(\oint f(z)\,dz\right)=\int_0^l dx\,[v(x,0)-v(x,h)]+\int_0^h dy\,[u(l,y)-u(0,y)]$$

$$v(x,0)-v(x,h)=-\int_0^h dy\,v_y(x,y) \qquad u(l,y)-u(0,y)=\int_0^l dx\,u_x(x,y)$$

$$=-\int_0^l dx\int_0^h dy\,v_y(x,y)+\int_0^h dy\int_0^l dx\,u_x(x,y)=0 \quad \text{since} \quad u_x=-v_y.$$

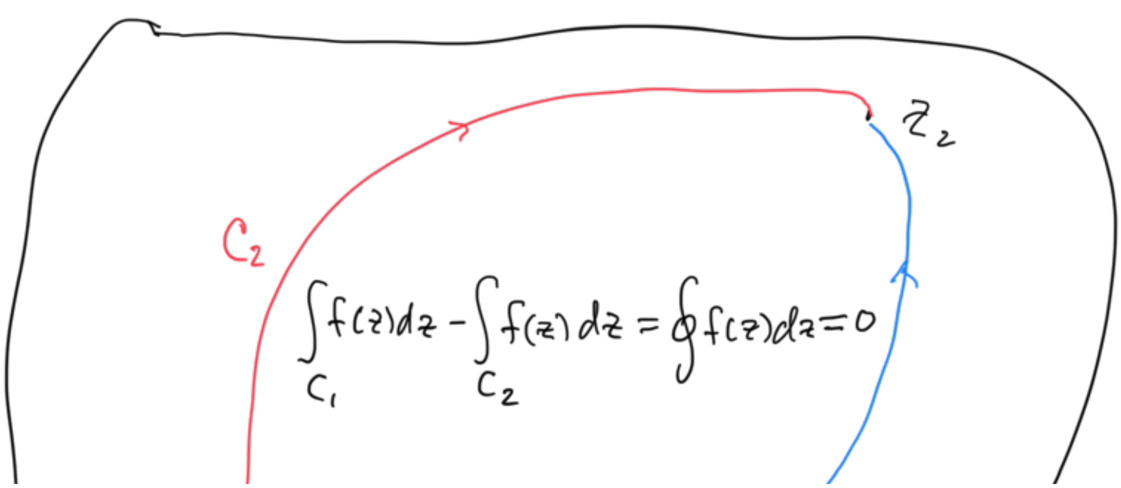
$$\oint_C f(z)\,dz=0$$



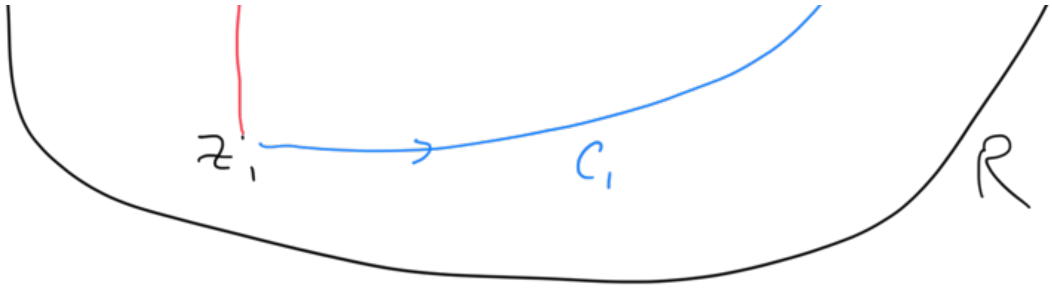


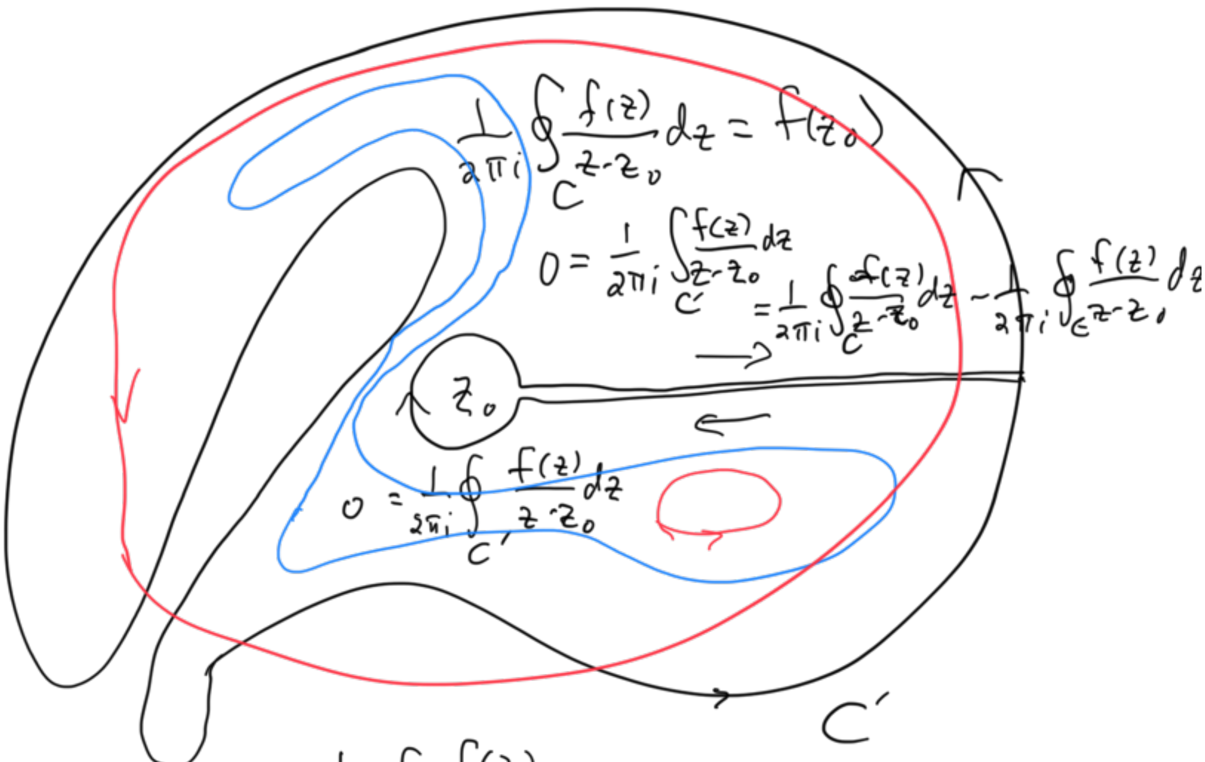
$$\oint_C f(z)\,dz = 0$$

if $f(z)$ analytic on and inside contour C.



$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = \oint f(z)\,dz = 0$$





$$0 = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz - f(z_0)$$

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0}}$$

$$f(z) = \oint \frac{f(z')}{(z'-z)}\frac{dz'}{(2\pi i)}$$

$$f'(z) = \oint \frac{f(z')}{2\pi i}\left(\frac{1}{z'-z-dz} - \frac{1}{z'-z}\right)\frac{dz'}{dz}$$

$$\frac{1}{\ } \int f(z') \left(z'-z-(z'-z-dz) \right] dz'$$

$$= \frac{1}{2\pi i} \oint f(z') \left(\frac{\quad}{(z'-z-dz)(z'-z)} \right) \frac{\quad}{dz}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z')}{(z'-z)^2}\, dz'$$

$$f''(z) = \frac{d}{dz} f'(z) = \frac{1}{2\pi i} \oint f(z') \frac{(-2)(-1)}{(z'-z)^3} dz'$$

$$= \frac{2}{2\pi i} \int \frac{f(z')\, dz'}{(z'-z)^3}$$

$$f'''(z) = \frac{2}{2\pi i} \frac{d}{dz} \int \frac{f(z')\, dz'}{(z'-z)^3} = \frac{2}{2\pi i} \int \frac{(-3)(-1) f(z')\, dz'}{(z'-z)^4}$$

$$= \frac{2 \cdot 3}{2\pi i} \oint \frac{f(z')\, dz'}{(z'-z)^4}$$

$$f^{(m)}(z) = \frac{m!}{2\pi i} \oint \frac{f(z')\, dz'}{(z'-z)^{m+1}}$$

$$ikx - L^2 (k-K)^2 = -L^2 (k - K + i\alpha x)^2 + \beta$$

$$= -L^2 (k-K)^2 - L^2\, 2 (k-K)\, i\alpha x + L^2 \alpha^2 x^2 + \beta$$

$$ik = -2i\alpha L^2 (k-K) \qquad \alpha = \frac{-k}{2L^2 (k-K)}$$

$$\beta = -L^2 \alpha^2 x^2$$

$$= \frac{-L^2 x^2 k^2}{4 L^4 (k-K)^2}$$

$$\psi(x,0) = N \int_{-\infty}^{\infty} dk\, e^{ikx - L^2 (k-K)^2}$$

$$ikx - L^2(k-K)^2 = -L^2(k-\alpha)^2 + \beta$$

$$= ikx - L^2k^2 + 2L^2Kk - L^2K^2 \,//$$

$$k(ix + 2L^2K) = 2L^2\alpha k$$

$$\alpha = \frac{ix + 2LK}{2L^2} \qquad -L^2K^2 = -L^2\alpha^2 + \beta$$

$$\beta = L^2(\alpha^2 - K^2)$$

$Lk = y$

$$\int_{-\infty}^{\infty} dk\, e^{-L^2(k-\alpha)^2+\beta} = e^{\beta}\int_{-\infty}^{\infty} dk\, e^{-L^2k^2} = \frac{e^{\beta}}{L}\int_{-\infty}^{\infty} dy\, e^{-y^2}$$

$$\langle x| e^{-iHt/\hbar} |\psi\rangle = \psi(x,t) = \langle x| e^{-\frac{iHt}{\hbar}} \int dp\, |p\rangle\langle p|\psi\rangle$$

$$\langle k|\psi\rangle = N e^{-L^2(k-K)^2}$$

$$H|p\rangle = \frac{p^2}{2m}|p\rangle \;\cancel{\#}\; \frac{\hbar^2k^2}{2m}|k\rangle = H|k\rangle$$

$$= \langle x| \int dp\, e^{-i\frac{p^2}{2m}t} |p\rangle\langle p|\psi\rangle$$

$$= N\int_{-\infty}^{\infty} dk\, e^{ikx - \frac{\hbar k^2 t}{2m} - L^2(k-K)^2}$$

$$ikx - \frac{\hbar k^2 t}{2m} - L^2(k-K)^2 = -\left(L^2 + \frac{\hbar t}{2m}\right)(k-\gamma)^2 + \delta$$

$$k(ix + 2L^2K) = 2k\gamma\left(L^2 + \frac{\hbar t}{2m}\right)$$

$$\gamma = \frac{ix + 2L^2K}{2\left(L^2 + \frac{\hbar t}{2m}\right)}$$

$$L^2K^2 = \delta - \gamma^2\left(L^2 + \frac{\hbar t}{}\right)$$

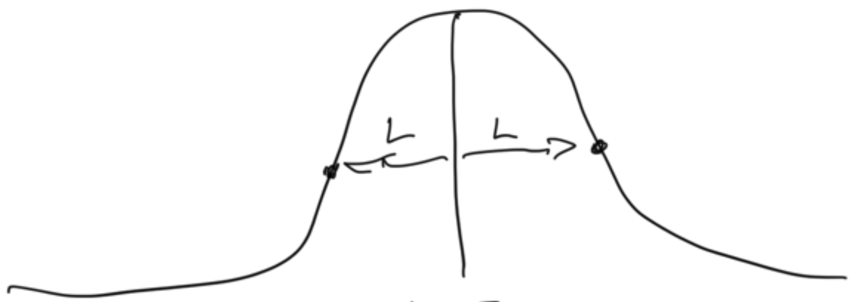
$$\int dk\, e^{-\left(L^2 + \frac{i\hbar t}{2m}\right)(k-q)^2 + \delta}$$

$$\psi(x,t) = \tilde{\mathcal{N}}\, e^{-\frac{(x - x(t))^2}{q^2(t)}}$$

$x = x(t)$

$|\tilde{\mathcal{N}}|^2$ independent of x.

$$\psi(x,t) = e^{-\frac{(x-\bar{x})^2}{L^2}} = e^{-\frac{(x - \bar{x}(t))^2}{(L(t))^2}}$$



$x = \bar{x}$

$$\psi(\bar{x} \pm L, t) = e^{-1} = \frac{1}{e} = \frac{1}{2.718281828}$$

$$(\Delta x)^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 |\psi(x,t)|^2 dx$$

$$\lim \frac{f(x',y') - f(x,y)}{x'-x+i(y'-y)} = f'(z)$$

$x' \to x$
$y' \to y$

$$df = f'(z)\,dz$$

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{z^{n+1}}$$

$|z| = R$

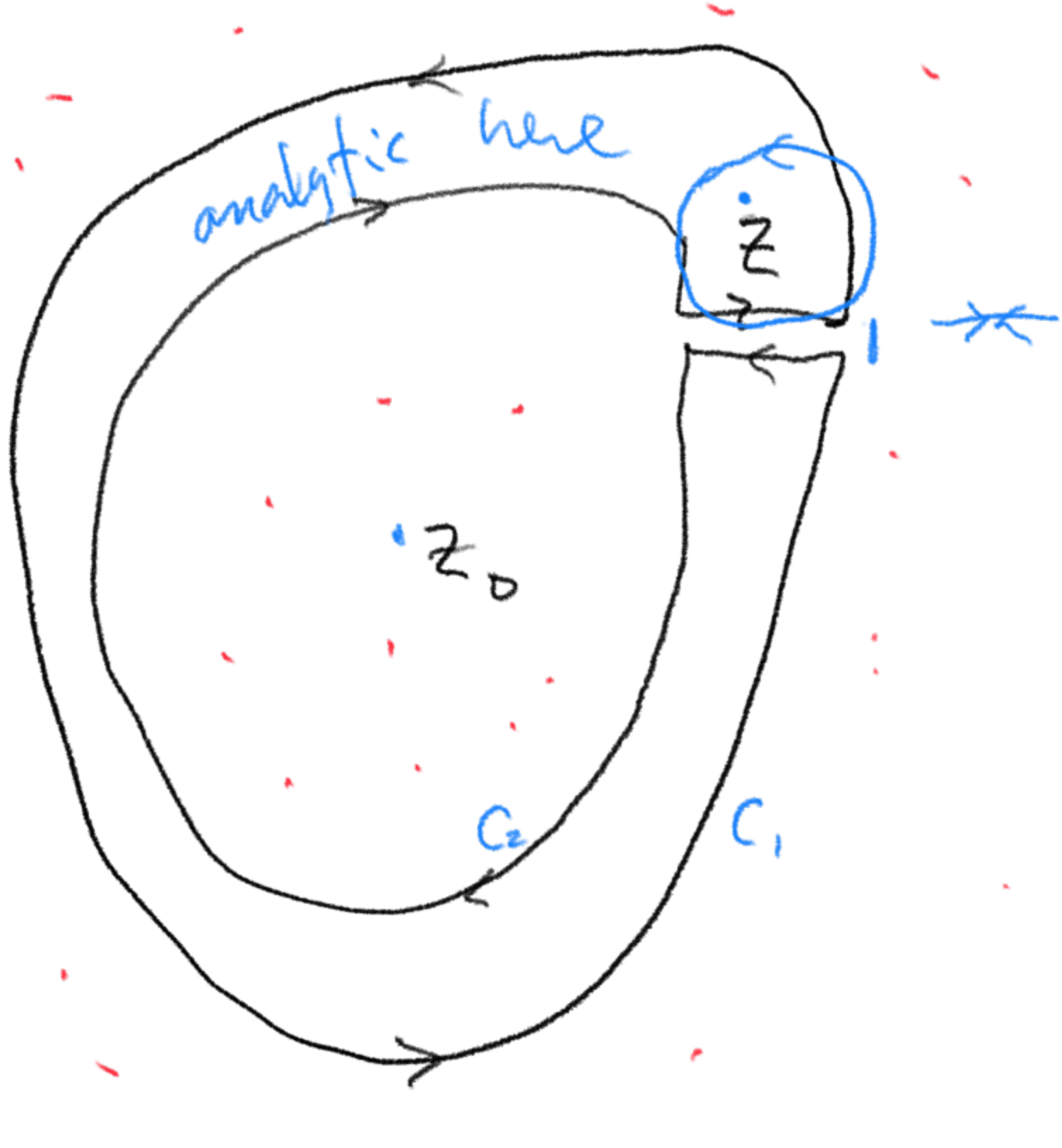
$|f(z)| \leq M$ on

$$\left| f^{(n)}(0) \right| \leq \frac{n!}{2\pi} \oint \frac{M R\, d\theta}{R^{n+1}}$$

$$\leq \frac{n!\ M}{}$$

$$= \frac{M \cdot}{2\pi} \frac{\quad}{R^n}$$

$$\leq M \sum \frac{r^n}{R^n} = \frac{M}{1 - \frac{r}{R}}$$



c_1 circles $r_2 > r_1$
c_2

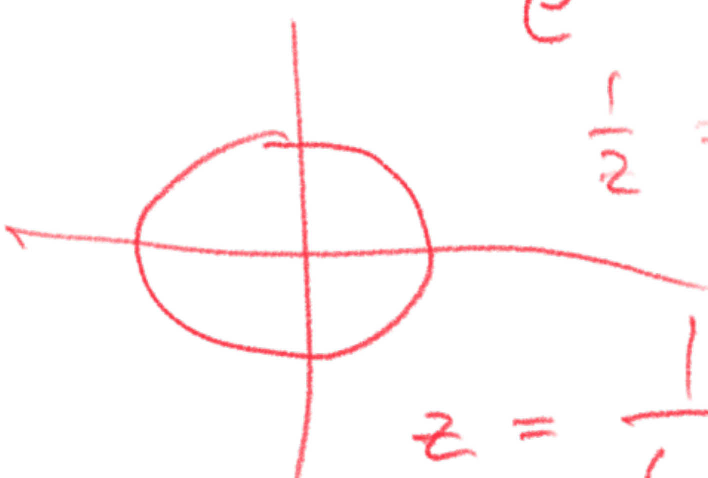
$$f(z) = \frac{1}{z-3}$$

$$f(z) = \frac{1}{z-(4-i)}$$

$e^{\frac{1}{z}}$ $e^{\frac{1}{z^2}}$

$$e^{\frac{1}{z}} = w$$

$$\frac{1}{z} = \log w$$



$$z = \frac{1}{\log w}$$

$f(z)$

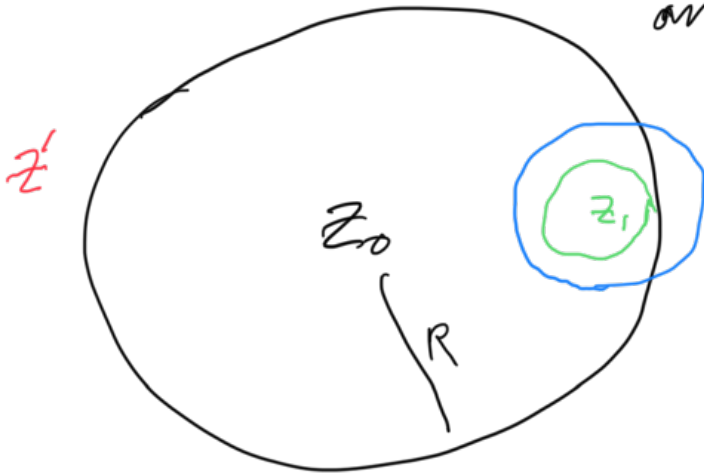
$\Gamma(z)$ analytic except at

$$\frac{1}{z-z_0}$$

$z = 0, -1, -2, \ldots$

$$\frac{1}{(z-z_0)^3}$$

analytic continuation



$\Gamma(z)$ is the archtype

$$I = \oint_{C_{-m}} a_{-m}(z_0)\,(z-z_0)^{-m}\,dz$$

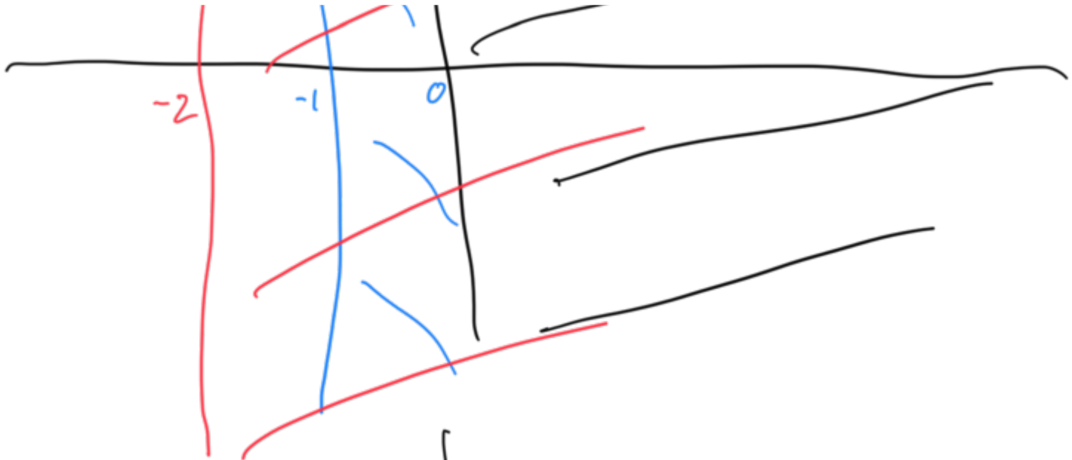
$$z = z_0 + \epsilon e^{i\theta}$$

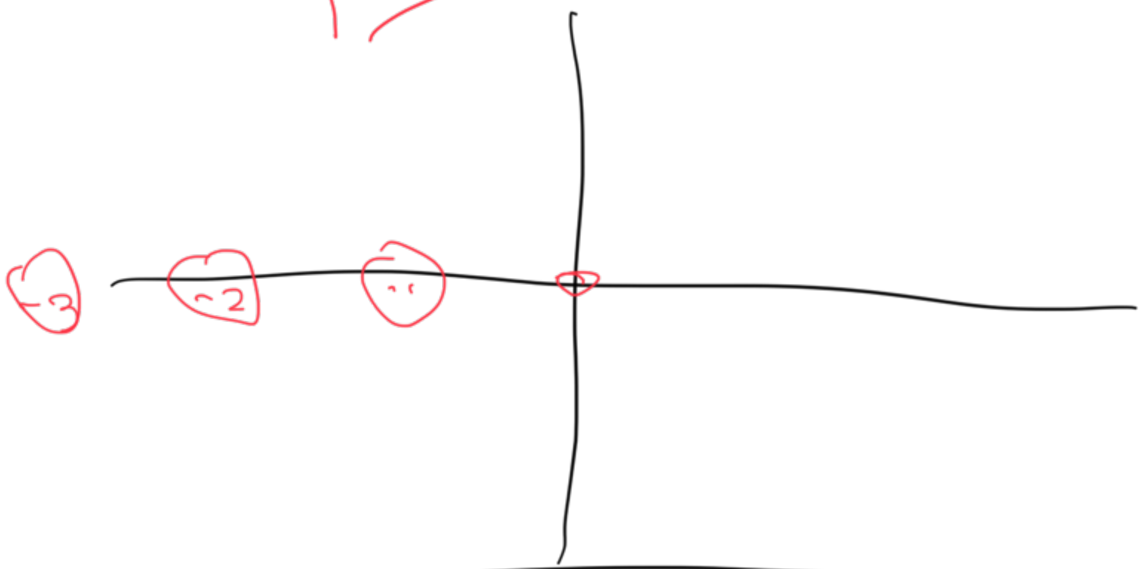
$$dz = i\epsilon e^{i\theta}\,d\theta$$

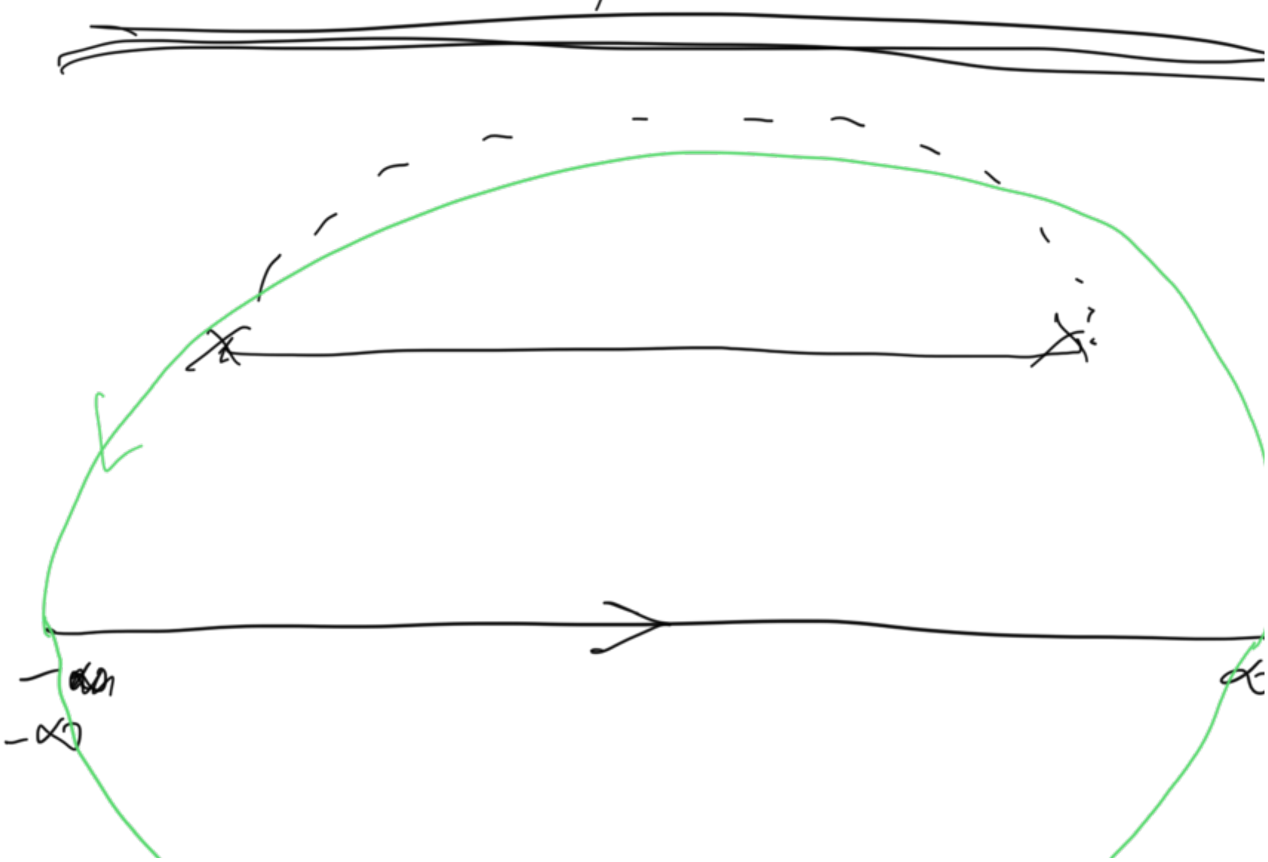
$$I = \int_0^{2\pi} a_{-m}(z_0)\,\frac{i\epsilon e^{i\theta}\,d\theta}{(\epsilon e^{i\theta})^m} = i\,\frac{a_{-m}(z_0)}{\epsilon^{m-1}}\int_0^{2\pi} e^{i\theta(1-m)}\,d\theta$$

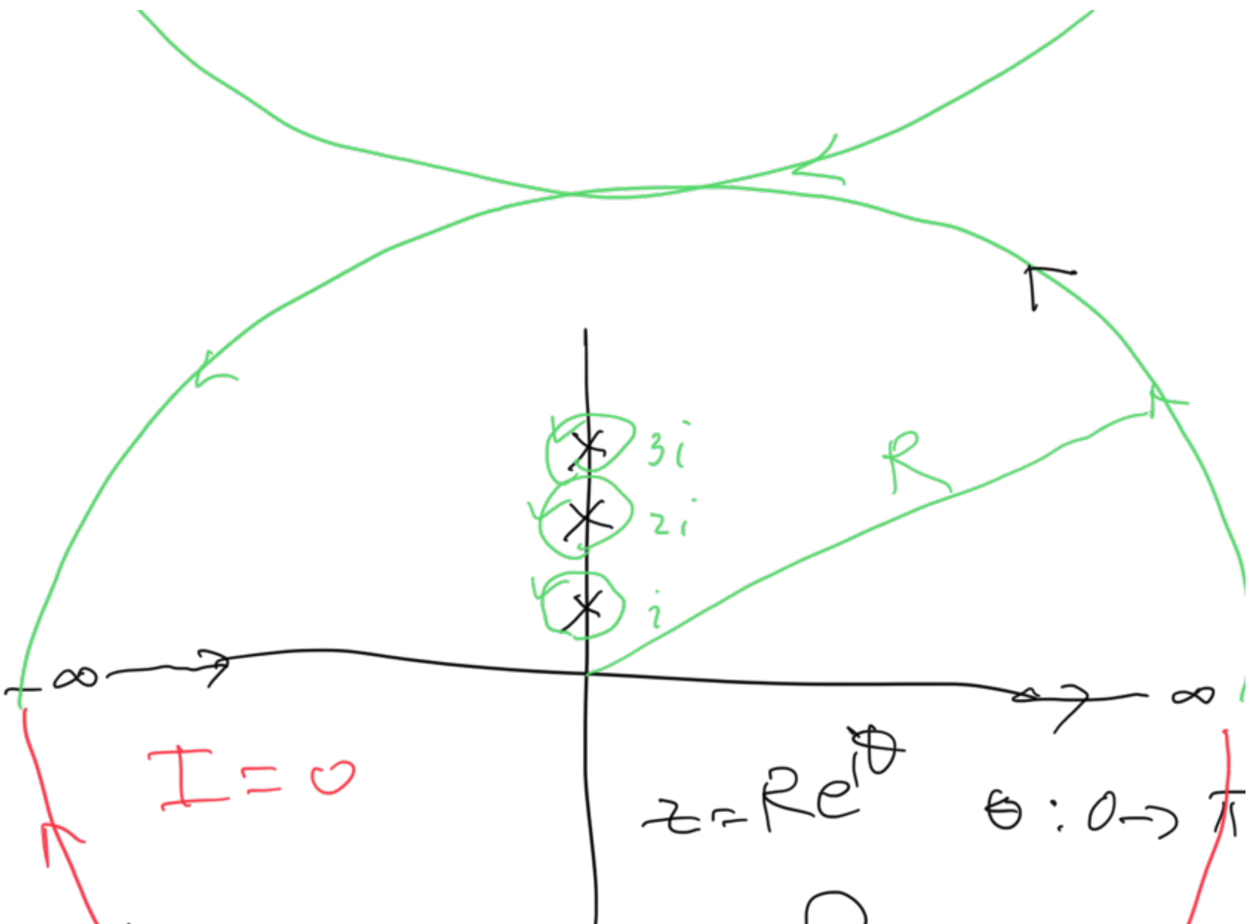
$$= \frac{i\,a_{-m}(z_0)}{\epsilon^{m-1}}\,2\pi\,\delta_{m,1} = 2\pi i\,a_{-1}(z_0)\,\delta_{m,-1}.$$

$$z\Gamma(z) = \cancel{\Gamma(z+1)}$$


-2
-1
0


-3
-2

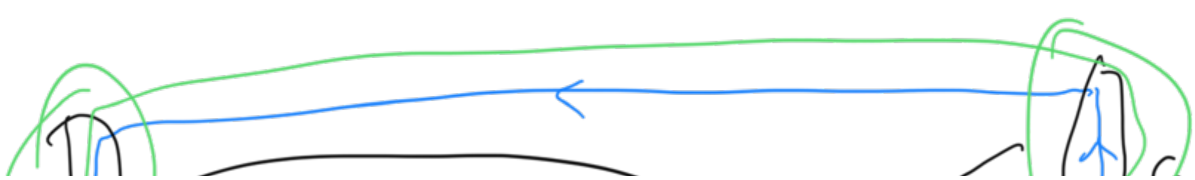

-∞
∞

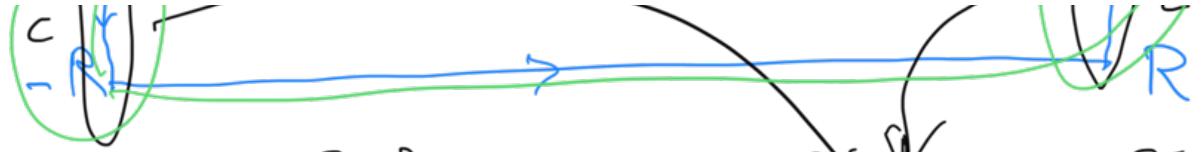


$$\left|\int_{\frown} f(z)\,dz\right| \simeq \frac{2\pi R}{R^3} \Rightarrow 0$$

$$f(z) = \frac{1}{(z-i)(z-2i)(z-3i)}$$

$$\int_{-\infty}^{\infty} e^{-m^2x^2}\,dx$$





$$0 = \oint dz\, e^{-m^2 z^2} \approx 0 \qquad c\, e^{-m^2 R^2} \to 0$$

$$\int dx\, e^{-m^2 x^2} = \int dx\, e^{-m^2 (x+ic)^2}$$

$$= \frac{\sqrt{\pi}}{m}$$

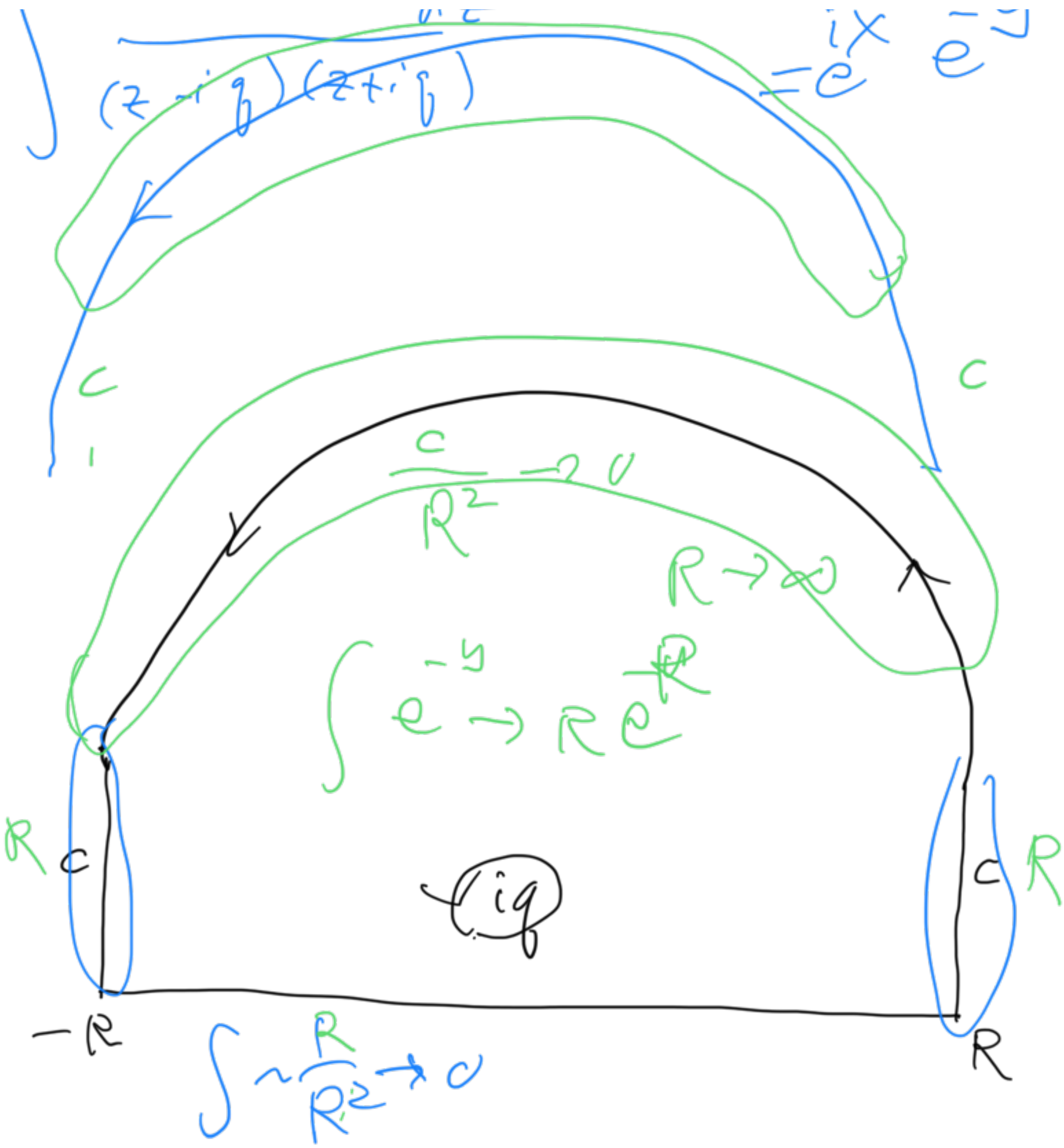
$$\tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ikx - m^2 x^2}$$

$$= \int \frac{dx}{\sqrt{2\pi}}\, e^{-m^2 (x+ic)^2 - m^2 c^2} \qquad c = \frac{k}{2m^2}$$

$$= e^{-m^2 c^2} \int \frac{dx}{\sqrt{2\pi}}\, e^{-m^2 (x+ic)^2} = e^{-m^2 c^2} \int \frac{dx}{\sqrt{2\pi}}\, e^{-m^2 x^2}$$

$$= \frac{e^{-m^2 c^2}}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{m} = \frac{1}{m\sqrt{2}}\, e^{-m^2 \left(\frac{k}{2m^2}\right)^2} = \frac{e^{-\frac{k^2}{4m^2}}}{m\sqrt{2}}$$

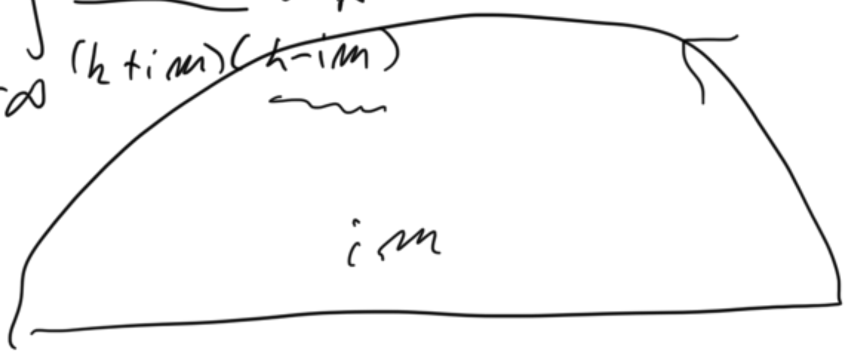
$$\int \quad e^{iz} \xrightarrow{dz} e^{i(x+iy)}$$


(z-iq)(z+iq)
= e^{ix} e^{-y}
c
c
c/R^2 → 0
R → ∞
∫ e^{-y} → Re^{-R}
R
c
iq
c
R
-R
R
∫ ~ R/R^2 → 0


-R
R
R
R
-iq

$$\int \frac{e^{-iz}}{(z-iq)(z+iq)}\, dz \qquad e^{-iz} = e^{-i(x+iy)} = e^{-ix} e^{y}$$

$$I(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{(k+im)(k-im)}\, dk \qquad k = x+iy$$

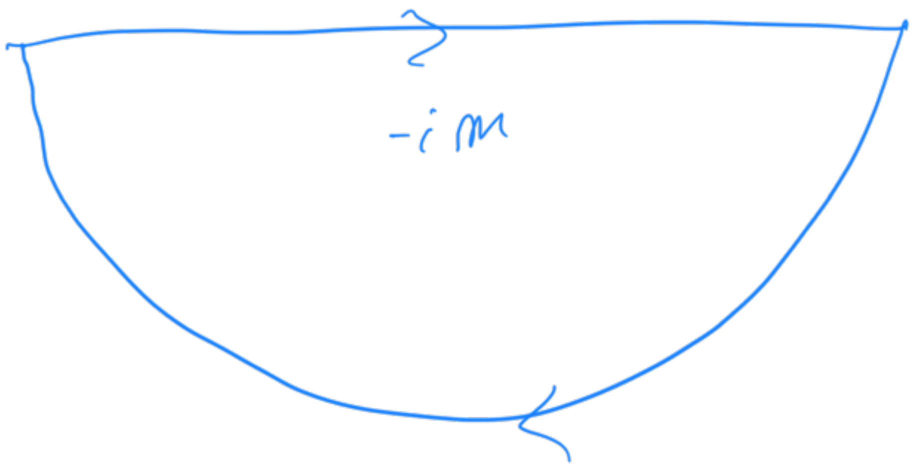


$x > 0$

$$I(x) = \frac{e^{-mx}}{2im} 2\pi i = \frac{\pi}{m} e^{-mx}$$

$x < 0$

$$I(x) = \int \frac{e^{ikx}\, dk}{(k+im)(k-im)}$$



$$= \frac{2\pi i \, e^{mx}}{-2im} = +\frac{\pi}{m} e^{mx}$$

$$I(x) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikx}}{k^2+m^2} = \frac{\pi}{m} e^{-m|x|} .$$

$$G_Y(r) = \frac{1}{ir} \oint \frac{dk}{(2\pi)^2} \frac{k}{\underline{(k-im)}\,\underset{2im}{(k+im)}} e^{ikr}$$

$-mr$

$$= 2\pi i \frac{1}{i r (2\pi)^2} \frac{im}{2im} e^{\cdots}$$

$$= \frac{e^{-mr}}{4\pi r}$$ Yukawa potential.

$$\frac{mrc^2}{\hbar c} \qquad e^{-mrc/\hbar}$$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$(n+\tfrac{1}{2})\hbar\omega$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a|0\rangle = 0$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a^\dagger a|n\rangle = a^\dagger \sqrt{n}\,|n-1\rangle = n|n\rangle$$

$$H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad \alpha \text{ complex}$$

$$\cdots = e^{-\frac{1}{2}|\alpha|^2} \sum^{\infty} \frac{\alpha^n}{\cdots} \sqrt{n}\,|n-1\rangle$$

$a|\alpha\rangle = e \quad \sum_{n=0} \quad \sqrt{n!}$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= \alpha\, e^{-\frac{1}{2}|\alpha|^2} \sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} |n-1\rangle \qquad m = n-1$$

$$= \alpha\, e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$$

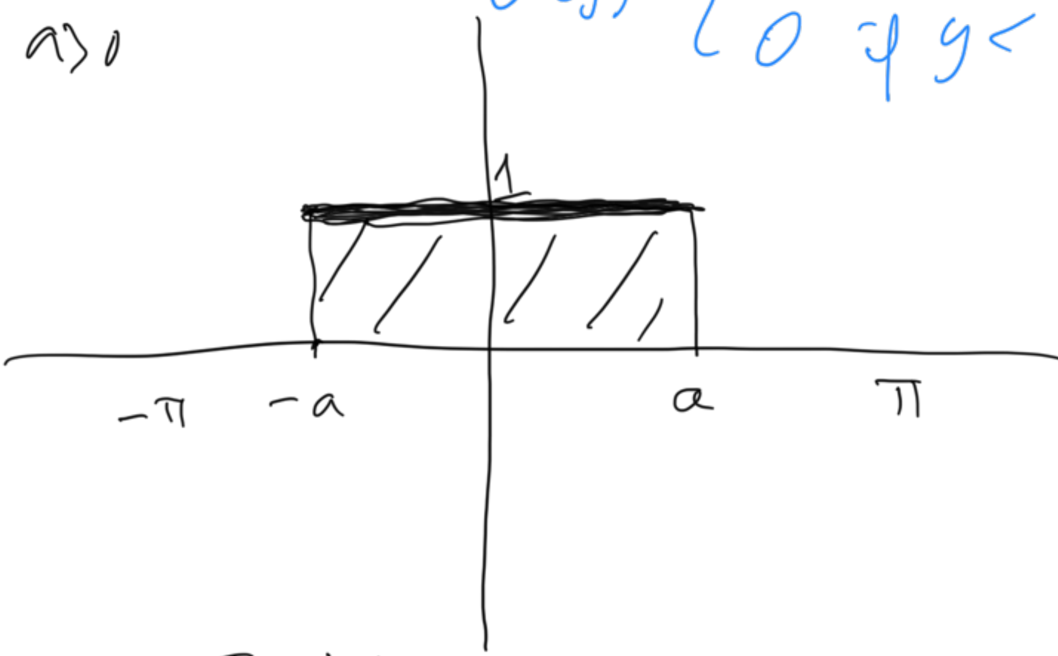
$$= \alpha |\alpha\rangle .$$

$$f(x) = \theta(a^2 - x^2) \qquad [-\pi, \pi]$$

$a^2 < \pi^2$

$$\theta(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

$a > 0$



$$f_n = \int^{\pi} e^{-inx}\, \theta(a^2 - x^2)\, dx$$

$$-\pi \sqrt{2\pi}$$

$$= \int_{-a}^{a} \frac{e^{-inx}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-inx}}{-in} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{-ina}}{-in} - \frac{e^{ina}}{-in} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{n} \left(\frac{e^{ina} - e^{-ina}}{2i} \right) 2$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{n} \sin na$$

$$f(x) = \theta(a^2 - x^2) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{inx}}{\sqrt{2\pi}}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{n\pi} \sin na \; e^{inx}$$

$$E_N(a_1, a_2 \cdots b_1, b_2 \cdots) \qquad 2N$$

$$\frac{\partial E_N}{\partial a_k} = 0 \text{ and } \frac{\partial E_N}{\partial b_\ell} = 0$$

$2N$ equations for $2N$ a_i, b_i.

$$\frac{\partial E_N}{\partial a_k} = \int_0^{2\pi} dx \left[f(x) - f_N(x)\right] \frac{\partial f_N(x)}{\partial a_k}$$

$$= \int_0^{2\pi} dx \left[f(x) - f_N(x)\right] \cos kx = 0$$

$$\frac{\partial E_N}{\partial b_\ell} = \int_0^{2\pi} \left[f(x) - f_N(x)\right] \sin \ell x \, dx = 0$$

$$\frac{\partial E_N}{\partial a_0} = \int_0^{2\pi} dx \left[f(x) - f_N(x)\right] \frac{1}{2} = 0$$

$$\int_0^{2\pi} dx\, f(x) = \int_0^{2\pi} dx\, f_N(x) = \frac{a_0}{2} 2\pi = \pi a_0$$

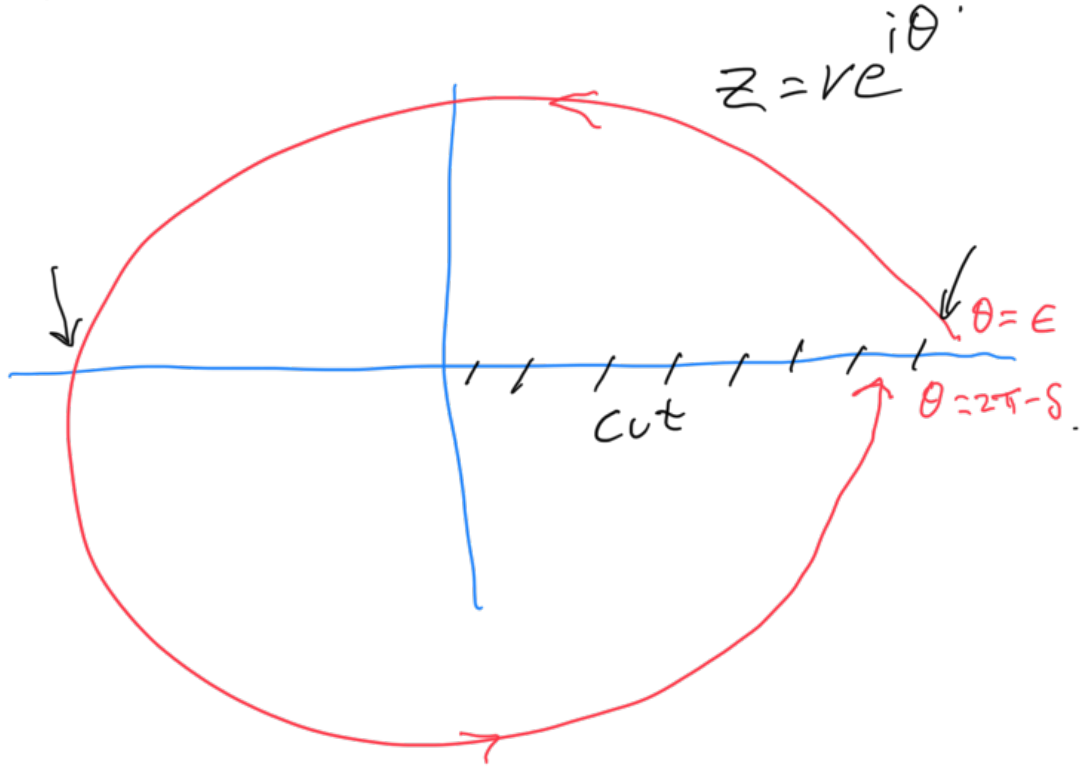
$$a_0 = \frac{1}{\pi} \int_0^{2\pi} dx\, f(x).$$

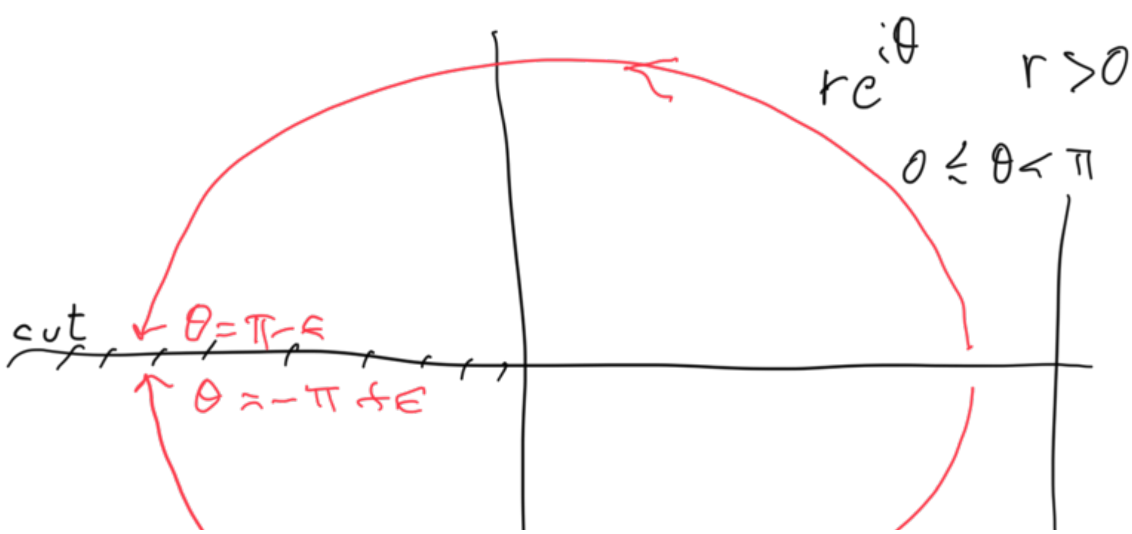
$$\int_0^{2\pi} dx\, f(x) \cos kx = \int_0^{2\pi} dx\, f_N(x) \cos kx = \pi a_k$$

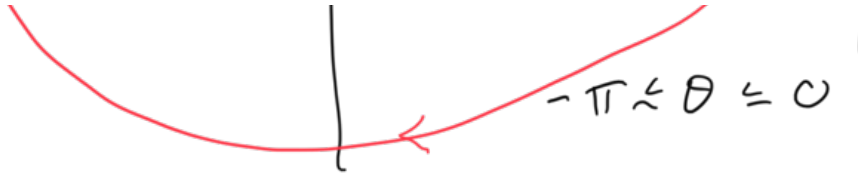
$$a_k = \frac{1}{\pi} \int_0^{2\pi} dx \; f(x) \cos kx$$

$$b_\ell = \frac{1}{\pi} \int_0^{2\pi} dx \; f(x) \sin \ell x$$

Two conventions:

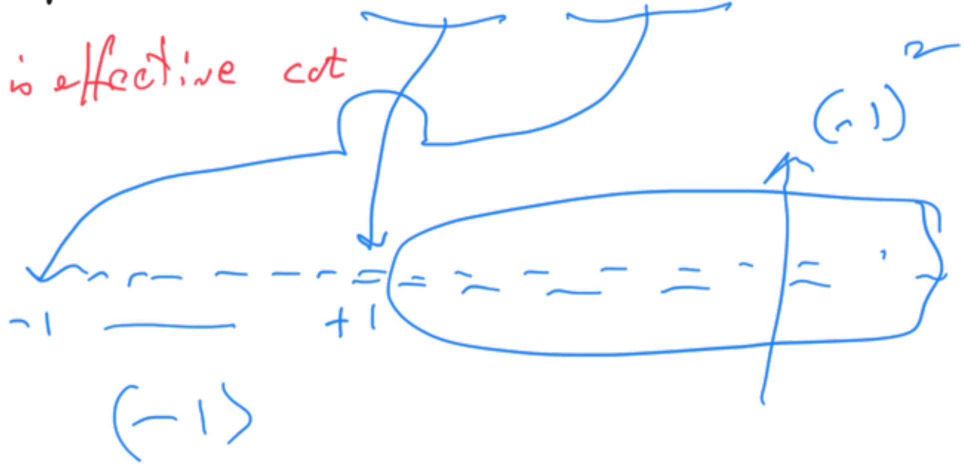


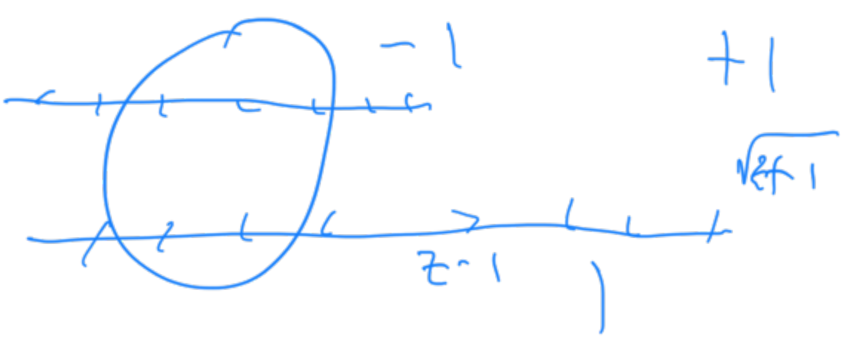




$$f(z) = \sqrt{z^2 - 1} = \sqrt{z-1}\,\sqrt{z+1}$$

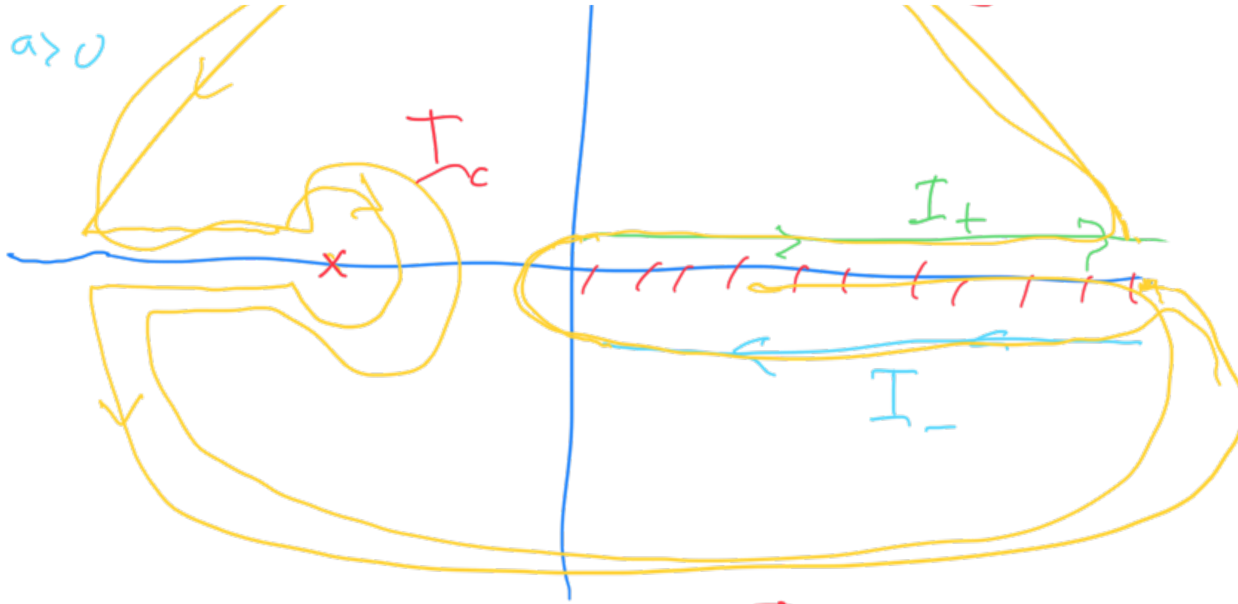
[-1, 1] is effective cut





$$I_s = \int_0^\infty \frac{dx}{(x+a)^2\sqrt{x}} = I_+ = I_-$$

I_{tot}



$$0 = I_s + I_- + \underset{\substack{\| \\ 0}}{I_{G+}} + I_c + \underset{\substack{\| \\ 0}}{I_{G-}}$$

$$2I_s = -I_c$$

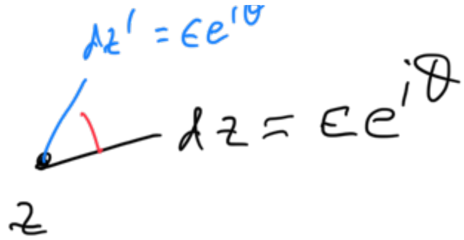
$$I_s = -\frac{1}{2} I_c = \frac{1}{2} \oint_{ccw} \frac{dz}{(z+a)^2\sqrt{z}}$$

$$= \pi i \frac{d}{dz} z^{-\frac{1}{2}} = -\frac{1}{2}\pi i\, z^{-3/2}$$

$$= -\frac{1}{2}\pi i \frac{1}{(-a)^{3/2}} = \frac{\pi i}{2a}\frac{1}{\sqrt{-a}} = \frac{\pi}{2\, a^{3/2}}$$

conformal mapping

$z \to f(z)$ f analytic



$\theta' - \theta = \Delta\theta$

$\lfloor z$

$$\frac{dz'}{dz} = \frac{\epsilon e^{i\theta'}}{\epsilon e^{i\theta}} = e^{i(\theta' - \theta)}$$

$\lfloor w$

$w = f(z)$

$$dw = f(z + dz) - f(z) = f'(z)\, dz$$

$$dw' = f(z + dz') - f(z) = f'(z)\, dz'$$

$$\frac{e^{i\phi'}}{e^{i\phi}} = \frac{dw'}{dw} = \frac{f'(z)\, dz'}{f'(z)\, dz} = \frac{dz'}{dz} = e^{i(\theta' - \theta)}$$

if $f'(z) \neq 0$, then angles are the same.

$$\phi' - \phi = \theta' - \theta$$

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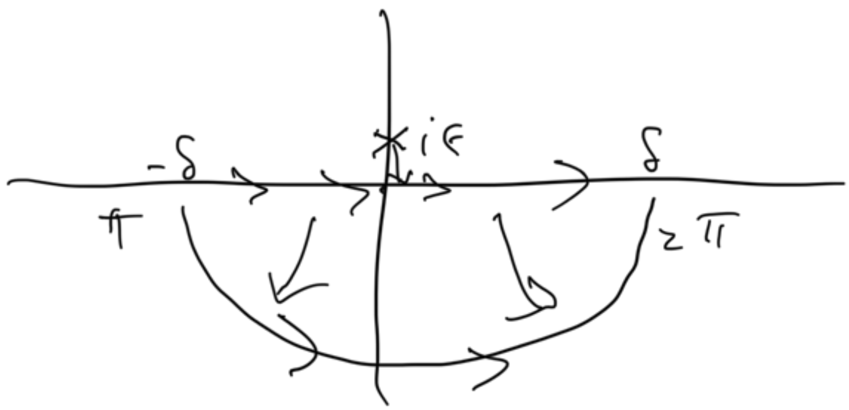
if $f'(z) = 0$

$$dw' = \tfrac{1}{2} f''(z)\, dz'^2 \simeq f(z + dz') - f(z)$$

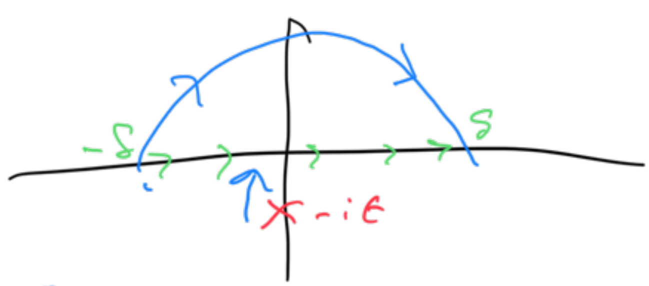
$$dw = \tfrac{1}{2} f''(z)\, dz^2 = f(z + dz) - f(z)$$

$$\frac{dw'}{dw} = \frac{dz^{2\prime}}{dz^2} = \frac{\epsilon^2 e^{2i\theta'}}{\epsilon^2 e^{2i\theta}} = e^{2i(\theta'-\theta)}$$

$$\delta > \epsilon > 0$$



$$f(0)\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} dz \frac{1}{z+i\epsilon}$$



$$z = \delta e^{i\theta}$$

$$dz = i\delta e^{i\theta} d\theta$$

$$f(0)\int_{\pi}^{0} d\theta \frac{i\delta e^{i\theta} d\theta}{\delta e^{i\theta}}$$

$$= f(0)\int_0^{\pi} (-1)\, d\theta\, i = -i\pi f(0)$$

$$\frac{1}{x+i\epsilon} = P\frac{1}{x} - i\pi\delta(x)$$

$$\int \frac{f(x)}{x+i\epsilon}dx = P\int \frac{f(x)dx}{x} - i\pi\int f(x)dx\delta(x)$$

$$= P\int \frac{f(x)}{x}dx - i\pi\delta(0)$$

$$\frac{1}{x-y\pm i\epsilon} = P\frac{1}{x-y} \mp i\delta(x-y)$$

$$P\frac{1}{k} = \frac{1}{k+i\epsilon} + i\delta(k)$$