2

Vector calculus

2.1 Derivatives and partial derivatives

The **derivative** of a function $f(x)$ at a point x is the limit of the ratio

$$\frac{df(x)}{dx} = \lim_{x' \to x} \frac{f(x') - f(x)}{x' - x}. \tag{2.1}$$

Example 2.1 (Derivative of a monomial) Setting $x' = x + \epsilon$ and letting $\varepsilon \to 0$, we compute the derivative of x^n as

$$\frac{dx^n}{dx} = \lim_{\epsilon \to 0} \frac{(x+\epsilon)^n - x^n}{\epsilon} \approx \frac{x^n + \epsilon\, n\, x^{n-1} - x^n}{\epsilon} = n\, x^{n-1}. \tag{2.2}$$

Similarly, adding fractions, we find

$$\frac{dx^{-n}}{dx} = \lim_{\epsilon \to 0} \frac{(x+\epsilon)^{-n} - x^{-n}}{\epsilon} \approx \frac{x^n - (x^n + \epsilon n x^{n-1})}{\epsilon x^{2n}} = -n\, x^{-n-1}. \tag{2.3}$$

□

The **partial derivative** of a function with respect to a given variable is the whole derivative of the function with its other variables held constant. For instance, the partial derivatives of the function $f(x,y,z) = x^\ell\, y^n / z^m$ with respect to x and z are

$$\frac{\partial f(x,y,z)}{\partial x} = \ell\, \frac{x^{\ell-1}\, y^n}{z^m} \quad \text{and} \quad \frac{\partial f(x,y,z)}{\partial z} = -\, m\, \frac{x^\ell\, y^n}{z^{m+1}}. \tag{2.4}$$

One often uses primes or dots to denote derivatives as in

$$f' = \frac{df}{dx}, \quad f'' = \frac{d^2 f}{dx^2} \equiv \frac{d}{dx}\left(\frac{df}{dx}\right), \quad \dot{f} = \frac{df}{dt}, \quad \text{and} \quad \ddot{f} = \frac{d^2 f}{dt^2}. \tag{2.5}$$

For higher or partial derivatives, one sometimes uses superscripts

$$f^{(k)} = \frac{d^k f}{dx^k} \quad \text{and} \quad f^{(k,\ell)} = \frac{\partial^{k+\ell} f}{\partial x^k \partial y^\ell} \tag{2.6}$$

or subscripts, sometimes preceded by commas

$$f_x = f_{,x} = \frac{\partial f}{\partial x} \quad \text{and} \quad f_{xyy} = f_{,xyy} = \frac{\partial^3 f}{\partial x \partial y^2}. \tag{2.7}$$

If variables $x = x_1, \ldots, x_n$ are labeled by indexes, derivatives can be labeled by subscripted indexes, sometimes preceded by commas

$$f_{,k} = \partial_k f = \frac{\partial f}{\partial x_k} \quad \text{and} \quad f_{,k\ell} = \partial_k \partial_\ell f = \frac{\partial^2 f}{\partial x_k \partial x_\ell}. \tag{2.8}$$

2.2 Gradient

The change $\boldsymbol{dp}$ in a point $\boldsymbol{p}$ due to changes du_1, du_2, du_3 in its **orthogonal** coordinates u_1, u_2, u_3 is a linear combination

$$\begin{aligned} \boldsymbol{dp} &= \frac{\partial \boldsymbol{p}}{\partial u_1}\, du_1 + \frac{\partial \boldsymbol{p}}{\partial u_2}\, du_2 + \frac{\partial \boldsymbol{p}}{\partial u_3}\, du_3 \\ &= \boldsymbol{e}_1\, du_1 + \boldsymbol{e}_2\, du_2 + \boldsymbol{e}_3\, du_3 \end{aligned} \tag{2.9}$$

of vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ that are orthogonal

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_k = h_i\, h_k\, \delta_{ik}. \tag{2.10}$$

In terms of the orthonormal vectors $\hat{\boldsymbol{e}}_j = \boldsymbol{e}_j / h_j$, the change $\boldsymbol{dp}$ is

$$\boldsymbol{dp} = h_1\, \hat{\boldsymbol{e}}_1\, du_1 + h_2\, \hat{\boldsymbol{e}}_2\, du_2 + h_3\, \hat{\boldsymbol{e}}_3\, du_3. \tag{2.11}$$

The orthonormal vectors $\hat{\boldsymbol{e}}_j$ have cyclic cross products

$$\hat{\boldsymbol{e}}_i \times \hat{\boldsymbol{e}}_j = \sum_{k=1}^{3} \epsilon_{ijk}\, \hat{\boldsymbol{e}}_k \tag{2.12}$$

in which ϵ_{ijk} is the antisymmetric Levi-Civita symbol (1.200) with $\epsilon_{123} = 1$.

In rectangular coordinates, the change $\boldsymbol{dp}$ in a physical point $\boldsymbol{p}$ due to changes dx, dy, and dz in its coordinates is $\boldsymbol{dp} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz$, and the scale factors are all unity $h_x = h_y = h_z = 1$. In cylindrical coordinates, the change $\boldsymbol{dp}$ in a point $\boldsymbol{p}$ due to changes $d\rho$, $d\phi$, and dz in its coordinates is $\boldsymbol{dp} = \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\phi}}\, d\phi + \hat{\boldsymbol{z}}\, dz$, and the scale factors are $h_\rho = 1$, $h_\phi = \rho$, and $h_z = 1$. In spherical coordinates, the change is $\boldsymbol{dp} = \hat{\boldsymbol{r}}\, dr + r\, \hat{\boldsymbol{\theta}}\, d\theta + r\, \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi$, and the scale factors are $h_r = 1$, $h_\theta = r$, and $h_\phi = r\, \sin\theta$. In these orthogonal coordinates, the change in a point is

$$\boldsymbol{dp} = \begin{cases} \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz \\ \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\phi}}\, d\phi + \hat{\boldsymbol{z}}\, dz \\ \hat{\boldsymbol{r}}\, dr + r\, \hat{\boldsymbol{\theta}}\, d\theta + r\, \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \end{cases}. \tag{2.13}$$

The **gradient** $\boldsymbol{\nabla} f$ of a scalar function f is defined so that its dot product $\boldsymbol{\nabla} f \cdot \boldsymbol{dp}$ with the change $\boldsymbol{dp}$ in the point $\boldsymbol{p}$ is the change df in f

$$\begin{aligned}\boldsymbol{\nabla} f \cdot \boldsymbol{dp} &= (\boldsymbol{\nabla} f_1\, \hat{\boldsymbol{e}}_1 + \boldsymbol{\nabla} f_2\, \hat{\boldsymbol{e}}_2 + \boldsymbol{\nabla} f_3\, \hat{\boldsymbol{e}}_3) \cdot (\hat{\boldsymbol{e}}_1 h_1 du_1 + \hat{\boldsymbol{e}}_2 h_2 du_2 + \hat{\boldsymbol{e}}_3 h_3 du_3) \\ &= \boldsymbol{\nabla} f_1\, h_1 du_1 + \boldsymbol{\nabla} f_2\, h_2 du_2 + \boldsymbol{\nabla} f_3\, h_3 du_3 \\ &= df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3. \end{aligned} \tag{2.14}$$

Thus the gradient in orthogonal coordinates is

$$\boldsymbol{\nabla} f = \frac{\hat{\boldsymbol{e}}_1}{h_1}\frac{\partial f}{\partial u_1} + \frac{\hat{\boldsymbol{e}}_2}{h_2}\frac{\partial f}{\partial u_2} + \frac{\hat{\boldsymbol{e}}_3}{h_3}\frac{\partial f}{\partial u_3}, \tag{2.15}$$

and in rectangular, cylindrical, and spherical coordinates it is

$$\boldsymbol{\nabla} f = \begin{cases} \hat{\boldsymbol{x}}\dfrac{\partial f}{\partial x} + \hat{\boldsymbol{y}}\dfrac{\partial f}{\partial y} + \hat{\boldsymbol{z}}\dfrac{\partial f}{\partial z} \\ \hat{\boldsymbol{\rho}}\dfrac{\partial f}{\partial \rho} + \dfrac{\hat{\boldsymbol{\phi}}}{\rho}\dfrac{\partial f}{\partial \phi} + \hat{\boldsymbol{z}}\dfrac{\partial f}{\partial z} \\ \hat{\boldsymbol{r}}\dfrac{\partial f}{\partial r} + \dfrac{\hat{\boldsymbol{\theta}}}{r}\dfrac{\partial f}{\partial \theta} + \dfrac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\dfrac{\partial f}{\partial \phi} \end{cases}. \tag{2.16}$$

In particular the gradient of $1/r$ is

$$\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2} \quad \text{and} \quad \boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{2.17}$$

In the last two formulas, the differentiation is with respect to r, not r'.

2.3 Divergence

The **divergence** of a vector $\boldsymbol{v}$ in an infinitesimal cube C is defined as the integral S of $\boldsymbol{v}$ over the surface of the cube divided by its volume $V = h_1 h_2 h_3\, du_1 du_2 du_3$. The surface integral S is the sum of the integrals of v_1, v_2, and v_3 over the cube's three forward faces $v_1 h_2 du_2 h_3 du_3 + v_2 h_1 du_1 h_3 du_3 + v_3 h_1 du_1 h_2 du_2$ minus the sum of the integrals of v_1, v_2, and v_3 over the cube's three opposite faces. The surface integral is then

$$S = \left[\frac{\partial(v_1 h_2 h_3)}{\partial u_1} + \frac{\partial(v_2 h_1 h_3)}{\partial u_2} + \frac{\partial(v_3 h_1 h_2)}{\partial u_3}\right] du_1 du_2 du_3. \tag{2.18}$$

So the divergence $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ is the ratio S/V

$$\boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{S}{V} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(v_1 h_2 h_3)}{\partial u_1} + \frac{\partial(v_2 h_1 h_3)}{\partial u_2} + \frac{\partial(v_3 h_1 h_2)}{\partial u_3}\right]. \tag{2.19}$$

In rectangular coordinates, the divergence of a vector $\boldsymbol{v} = (v_x, v_y, v_z)$ is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{2.20}$$

In cylindrical coordinates, it is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{1}{\rho}\left[\frac{\partial(v_\rho \rho)}{\partial \rho} + \frac{\partial v_\phi}{\partial \phi} + \frac{\partial(v_z \rho)}{\partial z}\right] = \frac{1}{\rho}\frac{\partial(\rho v_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}, \tag{2.21}$$

and in spherical coordinates it is

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = \frac{1}{r^2}\frac{\partial(v_r\, r^2)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(v_\theta\, \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}. \tag{2.22}$$

By assembling a suitable number of infinitesimal cubes, one may create a three-dimensional region of arbitrary shape and volume. The sum of the products of the divergence $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ in each cube times its volume dV is the sum of the surface integrals dS over the faces of these tiny cubes. The integrals over the interior faces cancel leaving just the integral over the surface ∂V of the whole volume V. Thus we arrive at Stokes's theorem

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{v}\, dV = \int_{\partial V} \boldsymbol{v} \cdot \boldsymbol{da} \tag{2.23}$$

in which $\boldsymbol{da}$ is an infinitesimal, outward, area element of the surface that is the boundary ∂V of the volume V.

Example 2.2 (Delta function) The integral of the divergence of the negative gradient $\hat{\boldsymbol{r}}/r^2$ (2.17) of $1/r$ over any sphere, however small, centered at the origin is 4π

$$\int \boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) dV = \int \frac{\hat{\boldsymbol{r}}}{r^2} \cdot \boldsymbol{da} = \int \frac{\hat{\boldsymbol{r}}}{r^2} \cdot r^2 \hat{\boldsymbol{r}}\, d\Omega = \int d\Omega = 4\pi. \tag{2.24}$$

Similarly, the integral of the divergence of $(\boldsymbol{r} - \boldsymbol{r}')/|\boldsymbol{r} - \boldsymbol{r}'|^3$ over any sphere, however small, centered at $\boldsymbol{r}'$ is 4π

$$\begin{aligned} \int \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right) dV &= \int \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \cdot \boldsymbol{da} \\ &= \int \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \cdot |\boldsymbol{r} - \boldsymbol{r}'|^2 \frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\, d\Omega = \int d\Omega = 4\pi. \end{aligned} \tag{2.25}$$

These divergences, vanishing for $\boldsymbol{r} \neq 0$ and for $\boldsymbol{r} - \boldsymbol{r}' \neq 0$, are delta functions

$$\boldsymbol{\nabla} \cdot \left(\frac{\hat{\boldsymbol{r}}}{\boldsymbol{r}^2}\right) = 4\pi\delta^3(\boldsymbol{r}) \quad \text{and} \quad \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{r} - \boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\right) = 4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}') \tag{2.26}$$

because if $f(\boldsymbol{r})$ is any suitably smooth function, then the integral over any volume that includes the point $\boldsymbol{r'}$ is

$$\int f(\boldsymbol{r})\, \boldsymbol{\nabla} \cdot \left(\frac{\boldsymbol{r} - \boldsymbol{r'}}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \right) d^3r = 4\pi\, f(\boldsymbol{r'}). \tag{2.27}$$

□

Example 2.3 (Gauss's law) The divergence $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ of the electric field is the charge density ρ divided by the electric constant $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}. \tag{2.28}$$

So by Stokes's theorem, the integral of the electric field over a surface ∂V that bounds a volume V is the charge inside divided by ϵ_0

$$\int_{\partial V} \boldsymbol{E} \cdot \boldsymbol{da} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{E}\, dV = \int_V \frac{\rho}{\epsilon_0}\, dV = \frac{Q_V}{\epsilon_0}. \tag{2.29}$$

□

2.4 Laplacian

The laplacian is the divergence (2.19) of the gradient (2.15). So in orthogonal coordinates it is

$$\triangle f \equiv \nabla^2 f \equiv \boldsymbol{\nabla} \boldsymbol{\cdot} \boldsymbol{\nabla} f = \frac{1}{h_1 h_2 h_3} \left[\sum_{k=1}^{3} \frac{\partial}{\partial u_k} \left(\frac{h_1 h_2 h_3}{h_k^2} \frac{\partial f}{\partial u_k} \right) \right]. \tag{2.30}$$

In rectangular coordinates, the laplacian is

$$\triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{2.31}$$

In cylindrical coordinates, it is

$$\triangle f = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 f}{\partial \phi^2} + \rho \frac{\partial^2 f}{\partial z^2} \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}, \tag{2.32}$$

and in spherical coordinates it is

$$\begin{aligned} \triangle f &= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\theta} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2} \\ &= \frac{1}{r} \frac{\partial^2}{\partial r^2} (r f) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}. \end{aligned} \tag{2.33}$$

Example 2.4 (Delta function as laplacian of $1/r$) By combining the gradient (2.17) of $1/r$ with the representation (2.26) of the delta function as a divergence, we can write delta functions as laplacians (with respect to $\boldsymbol{r}$)

$$-\triangle\left(\frac{1}{r}\right) = 4\pi\delta^3(\boldsymbol{r}) \quad \text{and} \quad -\triangle\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = 4\pi\delta^3(\boldsymbol{r}-\boldsymbol{r}'). \tag{2.34}$$

□

Example 2.5 (Electric field of a uniformly charged sphere) The electric field $\boldsymbol{E} = -\nabla\phi - \dot{\boldsymbol{A}}$ in static problems is just the gradient $\boldsymbol{E} = -\nabla\phi$ of the scalar potential ϕ. Gauss's law (2.28) then gives us Poisson's equation $\nabla \cdot \boldsymbol{E} = -\triangle\phi = \rho/\epsilon_0$. Writing the laplacian in spherical coordinates (2.33) and using spherical symmetry, we find

$$-\frac{1}{r}\frac{d^2}{dr^2}(r\phi) = \frac{\rho}{\epsilon_0} \tag{2.35}$$

in which ρ is the uniform charge density of the sphere. Integrating twice and letting the constant a be $\phi(r)$ at $r = 0$, we find the potential inside the sphere to be $\phi(r) = a - \rho\, r^2/(6\epsilon_0)$. Outside the sphere, the charge density vanishes, and so the second r-derivative of $r\phi$ vanishes, $(r\phi)'' = 0$. Integrating twice, we get for the potential outside the sphere $\phi(r) = b/r$ after dropping a constant term because $\phi(r) \to 0$ as $r \to \infty$. The interior and exterior solutions for the electric field $\boldsymbol{E} = -\nabla\phi$ agree where the meet on the surface of the sphere at $r = R$

$$\boldsymbol{E} = \hat{\boldsymbol{r}}\frac{\rho R}{3\epsilon_0} = \hat{\boldsymbol{r}}\frac{b}{R^2}. \tag{2.36}$$

Thus $b = \rho R^3/(3\epsilon_0)$. Matching the interior potential to the exterior potential on the surface of the sphere gives $a = \rho R^2/(2\epsilon_0)$. So the potential of a uniformly charged sphere of radius R is

$$\phi(r) = \begin{cases} \rho\left(R^2 - r^2/3\right)/(2\epsilon_0) & r \le R \\ \rho R^3/(3\epsilon_0 r) & r \ge R \end{cases}. \tag{2.37}$$

□

2.5 Curl

The directed area $\boldsymbol{dS}$ of an infinitesimal rectangle whose sides are the tiny perpendicular vectors $h_i\hat{\boldsymbol{e}}_i du_i$ and $h_j\hat{\boldsymbol{e}}_j du_j$ (fixed i and j) is their cross-product (2.12)

$$\boldsymbol{dS} = h_i \hat{\boldsymbol{e}}_i du_i \times h_j \hat{\boldsymbol{e}}_j du_j = \sum_{k=1}^{3} \epsilon_{ijk} \, \hat{\boldsymbol{e}}_k \, h_i h_j \, du_i du_j. \tag{2.38}$$

The line integral of the vector $\boldsymbol{f}$ along the perimeter of this infinitesimal rectangle is

$$\oint \boldsymbol{f} \cdot \boldsymbol{dl} = \left(\frac{\partial \left(h_j f_j \right)}{\partial u_i} - \frac{\partial \left(h_i f_i \right)}{\partial u_j} \right) du_i \, du_j. \tag{2.39}$$

The **curl** $\boldsymbol{\nabla} \times \boldsymbol{f}$ of a vector $\boldsymbol{f}$ is defined to be the vector whose dot product with the area (2.38) is the line integral (2.39)

$$(\boldsymbol{\nabla} \times \boldsymbol{f}) \cdot \boldsymbol{dS} = \left(\nabla \times f \right)_k h_i h_j du_i du_j = \left(\frac{\partial \left(h_j f_j \right)}{\partial u_i} - \frac{\partial \left(h_i f_i \right)}{\partial u_j} \right) du_i \, du_j \tag{2.40}$$

in which i, j, k are 1, 2, 3 or a cyclic permutation of 1, 2, 3. Thus the kth component of the curl is

$$(\nabla \times f)_k = \frac{1}{h_i h_j} \left(\frac{\partial (h_j f_j)}{\partial u_i} - \frac{\partial (h_i f_i)}{\partial u_j} \right) \quad \text{(no sum)}, \tag{2.41}$$

and the curl as a vector field is the sum over i, j, and k from 1 to 3

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} \frac{\hat{\boldsymbol{e}}_k}{h_i h_j} \frac{\partial (h_j f_j)}{\partial u_i}. \tag{2.42}$$

In rectangular coordinates, the scale factors are all unity, and the ith component of the curl $\boldsymbol{\nabla} \times \boldsymbol{f}$ is

$$(\nabla \times f)_i = \sum_{j,k=1}^{3} \epsilon_{ijk} \frac{\partial f_k}{\partial x_j} = \sum_{j,k=1}^{3} \epsilon_{ijk} \, \partial_j \, f_k. \tag{2.43}$$

We can write the curl as a determinant

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\boldsymbol{e}}_1 & h_2 \hat{\boldsymbol{e}}_2 & h_3 \hat{\boldsymbol{e}}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}. \tag{2.44}$$

In rectangular coordinates, the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{vmatrix}. \tag{2.45}$$

In cylindrical coordinates, it is

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \frac{1}{\rho} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \hat{\boldsymbol{z}} \\ \partial_\rho & \partial_\phi & \partial_z \\ f_\rho & \rho f_\phi & f_z \end{vmatrix} \tag{2.46}$$

and in spherical coordinates, it is

$$\boldsymbol{\nabla} \times \boldsymbol{f} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\boldsymbol{r}} & r\,\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ f_r & r\,f_\theta & r\sin\theta\, f_\phi \end{vmatrix}. \tag{2.47}$$

Sums of products of two Levi-Civita symbols (1.200) yield useful identities

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km} \quad \text{and} \quad \sum_{i,j=1}^{3} \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn} \tag{2.48}$$

in which δ_{jm} is Kronecker's delta (1.38). Thus the curl of a curl is

$$\begin{aligned} \big[\nabla \times (\nabla \times A)\big]_i &= \sum_{j,k,m,n=1}^{3} \epsilon_{ijk}\partial_j \epsilon_{kmn}\partial_m A_n = \sum_{j,m,n=1}^{3} (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\partial_j\partial_m A_n \\ &= \partial_i \nabla \cdot A - \triangle A_i \quad \text{or} \quad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \triangle \boldsymbol{A}. \end{aligned} \tag{2.49}$$

By assembling a suitable set of infinitesimal rectangles $\boldsymbol{dS}$, we may create an arbitrary surface S. The surface integral of the dot product $\boldsymbol{\nabla} \times \boldsymbol{f} \cdot \boldsymbol{dS}$ over the tiny rectangles $\boldsymbol{dS}$ that make up the surface S is the sum of the line integrals along the sides of these tiny rectangles. The line integrals over the interior sides cancel leaving just the line integral along the boundary ∂S of the finite surface S. Thus the integral of the curl $\boldsymbol{\nabla} \times \boldsymbol{f}$ of a vector $\boldsymbol{f}$ over a surface is the line integral of the vector $\boldsymbol{f}$ along the boundary of the surface

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{f}) \cdot \boldsymbol{dS} = \int_{\partial S} \boldsymbol{f} \cdot \boldsymbol{d\ell} \tag{2.50}$$

which is one of Stokes's theorems.

Example 2.6 (Maxwell's equations) In empty space, Maxwell's equations in SI units are $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0, \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$, and $c^2\, \boldsymbol{\nabla} \times \boldsymbol{B} = \dot{\boldsymbol{E}}$. They imply that the voltage induced in a loop is the negative of the rate of change of the magnetic induction through the loop

$$V = \oint_{\partial S} \boldsymbol{E} \cdot \boldsymbol{dx} = -\dot{\Phi}_B = -\int_S \dot{\boldsymbol{B}} \cdot \boldsymbol{da} \tag{2.51}$$

and that the magnetic induction induced in a loop is the rate of change of the electric flux through the loop divided by c^2

$$B = \int_{\partial S} \boldsymbol{B} \cdot \boldsymbol{dx} = \frac{1}{c^2}\,\dot{\Phi}_E = \frac{1}{c^2}\int_S \dot{\boldsymbol{E}} \cdot \boldsymbol{da}. \tag{2.52}$$

Maxwell's equations in empty space and the curl identity (2.49) imply that

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \triangle \boldsymbol{E} = -\triangle \boldsymbol{E} = -\boldsymbol{\nabla} \times \dot{\boldsymbol{B}} = -\ddot{\boldsymbol{E}}/c^2 \tag{2.53}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \triangle \boldsymbol{B} = -\triangle \boldsymbol{B} = \boldsymbol{\nabla} \times \dot{\boldsymbol{E}}/c^2 = -\ddot{\boldsymbol{B}}/c^2 \tag{2.54}$$

or

$$\triangle \boldsymbol{E} = \ddot{\boldsymbol{E}}/c^2 \quad \text{and} \quad \triangle \boldsymbol{B} = \ddot{\boldsymbol{B}}/c^2. \tag{2.55}$$

The exponentials $\boldsymbol{E}(\boldsymbol{k},\omega) = \boldsymbol{\epsilon}\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ and $\boldsymbol{B}(\boldsymbol{k},\omega) = (\hat{\boldsymbol{k}} \times \boldsymbol{\epsilon}/c)\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ with $\omega = |\boldsymbol{k}|c$ and $\hat{\boldsymbol{k}} \cdot \boldsymbol{\epsilon} = 0$ obey these wave equations. □

Example 2.7 Helmholtz decomposition

We can use the delta-function formula (2.34) to write any suitably smooth 3-dimensional vector field $\boldsymbol{V}(\boldsymbol{x})$ as

$$\boldsymbol{V}(\boldsymbol{x}) = -\int \boldsymbol{V}(\boldsymbol{r})\, \triangle \left(\frac{1}{|\boldsymbol{r}-\boldsymbol{x}|}\right) d^3r \tag{2.56}$$

in which the derivatives $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla}^2 = \triangle$ can be both with respect to $\boldsymbol{x}$ or both with respect to $\boldsymbol{r}$. Taking them to be with respect to $\boldsymbol{x}$, we have

$$\boldsymbol{V}(\boldsymbol{x}) = -\boldsymbol{\nabla}^2 \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{x}|}\, d^3r. \tag{2.57}$$

We now use our formula (2.49) for the curl of a curl

$$\boldsymbol{\nabla}^2 \boldsymbol{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{V}) \tag{2.58}$$

to write $\boldsymbol{V}(\boldsymbol{x})$ as

$$\boldsymbol{V}(\boldsymbol{x}) = -\boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{x}|}\, d^3r\right) + \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{x}|}\, d^3r\right). \tag{2.59}$$

Thus any suitably smooth 3-dimensional vector field $\boldsymbol{V}(\boldsymbol{x})$ can be written as the sum

$$\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{\nabla}\phi(\boldsymbol{x}) + \boldsymbol{\nabla} \times \boldsymbol{A}(\boldsymbol{x}) \tag{2.60}$$

of the gradient of a scalar field

$$\phi(\boldsymbol{x}) = -\boldsymbol{\nabla} \cdot \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{x}|}\, d^3r \tag{2.61}$$

and the curl of a vector field

$$\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{V}(\boldsymbol{r})}{|\boldsymbol{r} - \boldsymbol{x}|} \, d^3 r \tag{2.62}$$

(Hermann von Helmholtz, 1821–1894). □

Exercises

2.1 Derive the first of the Levi-Civita identities (2.48).

2.2 Derive the second of the Levi-Civita identities (2.48).

2.3 Use the first of the Levi-Civita identities (2.48) to show that every 3-vector $\boldsymbol{V}$ can be expressed in terms of any nonzero 3-vector $\boldsymbol{k}$ as

$$\boldsymbol{V} = \frac{1}{\boldsymbol{k} \cdot \boldsymbol{k}} \Big((\boldsymbol{k} \cdot \boldsymbol{V}) \, \boldsymbol{k} - \boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{V}) \Big). \tag{2.63}$$

2.4 Show that

$$\boldsymbol{\nabla} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \, \boldsymbol{\nabla} \cdot \boldsymbol{b} - \boldsymbol{b} \, \boldsymbol{\nabla} \cdot \boldsymbol{a} + (\boldsymbol{b} \cdot \boldsymbol{\nabla})\boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{\nabla})\boldsymbol{b}. \tag{2.64}$$

2.5 Simplify $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi$ and $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{a})$ in which ϕ is a scalar field and $\boldsymbol{a}$ is a vector field.

2.6 Simplify $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\phi \times \boldsymbol{\nabla}\psi)$ in which ϕ and ψ are scalar fields.

2.7 Let $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \dot{\boldsymbol{A}}$ and show that Maxwell's equations in vacuum (example 2.6) and the Lorentz gauge condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \dot{\phi}/c^2 = 0 \tag{2.65}$$

imply that $\boldsymbol{A}$ and ϕ obey the wave equations

$$\triangle\phi - \ddot{\phi}/c^2 = 0 \quad \text{and} \quad \triangle\boldsymbol{A} - \ddot{\boldsymbol{A}}/c^2 = 0. \tag{2.66}$$