Much is known about Legendre functions. The books *A Course of Modern Analysis* (Whittaker and Watson, 1927, chap. XV) and *Methods of Mathematical Physics* (Courant and Hilbert, 1955) are classics. The NIST Digital Library of Mathematical Functions (dlmf.nist.gov) and the companion *NIST Handbook of Mathematical Functions* (Olver et al., 2010) are outstanding. You can learn more about the CMB in Steven Weinberg's book *Cosmology* (Weinberg, 2010, chap. 7) and at the website http://camb.info.

Exercises

8.1 Use conditions (8.6) and (8.7) to find $P_0(x)$ and $P_1(x)$.

8.2 Using the Gram-Schmidt method (section 1.10) to turn the functions x^n into a set of functions $L_n(x)$ that are orthonormal on the interval $[-1,1]$ with inner product (8.2), find $L_n(x)$ for $n = 0$, 1, 2, and 3. Isn't Rodrigues's formula (8.8) easier to use?

8.3 Derive the conditions (8.6–8.7) on the coefficients a_k of the Legendre polynomial $P_n(x) = a_0 + a_1 x + \ldots + a_n x^n$.

8.4 Use equations (8.6–8.7) to find $P_3(x)$ and $P_4(x)$.

8.5 In superscript notation (6.19), Leibniz's rule (4.49) for derivatives of products $u\,v$ of functions is

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}\, v^{(k)}. \tag{8.128}$$

Use it and Rodrigues's formula (8.8) to derive the explicit formula (8.9).

8.6 The product rule for derivatives in superscript notation (6.19) is

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)}\, v^{(k)}. \tag{8.129}$$

Apply it to Rodrigues's formula (8.8) with $x^2 - 1 = (x-1)(x+1)$ and show that the Legendre polynomials satisfy $P_n(1) = 1$.

8.7 Use Cauchy's integral formula (5.41) and Rodrigues's formula (8.55) to derive Schlaefli's integral formula (8.56).

8.8 Show that the polynomials (8.57) are orthogonal (8.58) as long as they satisfy the endpoint condition (8.59).

8.9 Derive the orthogonality relation (8.2) from Rodrigues's formula (8.8).

8.10 (a) Use the fact that the quantities $w = x^2 - 1$ and $w_n = w^n$ vanish at the end points ± 1 to show by repeated integrations by parts that in superscript notation (6.19)

$$\int_{-1}^{1} w_n^{(n)} w_n^{(n)} dx = -\int_{-1}^{1} w_n^{(n-1)} w_n^{(n+1)} dx = (-1)^n \int_{-1}^{1} w_n w_n^{(2n)} dx. \tag{8.130}$$

(b) Show that the final integral is equal to

$$I_n = (2n)! \int_{-1}^{1} (1-x)^n \, (1+x)^n \, dx. \tag{8.131}$$

8.11 (a) Show by integrating by parts that $I_n = (n!)^2 \, 2^{2n+1}/(2n+1)$. (b) Prove (8.13).

8.12 Suppose that $P_n(x)$ and $Q_n(x)$ are two solutions of (8.27). Find an expression for their wronskian, apart from an overall constant.

8.13 Use the method of sections (6.25 & 6.32) and the solution $f(r) = r^\ell$ to find a second solution of the ode (8.83).

8.14 For a uniformly charged circle of radius a, find the resulting scalar potential $\phi(r, \theta)$ for $r < a$.

8.15 (a) Find the electrostatic potential $V(r, \theta)$ outside an uncharged perfectly conducting sphere of radius R **in** a vertical uniform static electric field that tends to $\boldsymbol{E} = E\hat{\boldsymbol{z}}$ as $r \to \infty$. (b) Find the potential if the free charge on the sphere is q_f.

8.16 Derive (8.126) from (8.124) and (8.125).

8.17 Find the electrostatic potential $V(r, \theta)$ inside a hollow sphere of radius R if the potential on the sphere is $V(R, \theta) = V_0 \cos^2 \theta$.

8.18 Find the electrostatic potential $V(r, \theta)$ outside a hollow sphere of radius R if the potential on the sphere is $V(R, \theta) = V_0 \cos^2 \theta$.