the basic vectors, rather than leave their lengths arbitrary, and so introduce a further stage of simplification into the representation. However, it is possible to normalize them only if the parameters which label them all take on discrete values. If any of these parameters are continuous variables that can take on all values in a range, the basic vectors are eigenvectors of some observable belonging to eigenvalues in a range and are of infinite length, from the discussion in § 10 (see p. 39 and top of p. 40). Some other procedure is then needed to fix the numerical factors by which the basic vectors may be multiplied. To get a convenient method of handling this question a new mathematical notation is required, which will be given in the next section.

15. The δ function

Our work in § 10 led us to consider quantities involving a certain kind of infinity. To get a precise notation for dealing with these infinities, we introduce a quantity $\delta(x)$ depending on a parameter x satisfying the conditions

$$\left.\begin{aligned} \int\limits_{-\infty}^{\infty} \delta(x)\,dx &= 1 \\ \delta(x) &= 0 \text{ for } x \neq 0. \end{aligned}\right\} \quad (2)$$

To get a picture of $\delta(x)$, take a function of the real variable x which vanishes everywhere except inside a small domain, of length ϵ say, surrounding the origin $x = 0$, and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations (for example provided the function is always of order ϵ^{-1}). Then in the limit $\epsilon \to 0$ this function will go over into $\delta(x)$.

$\delta(x)$ is not a function of x according to the usual mathematical definition of a function, which requires a function to have a definite value for each point in its domain, but is something more general, which we may call an 'improper function' to show up its difference from a function defined by the usual definition. Thus $\delta(x)$ is not a quantity which can be generally used in mathematical analysis like an ordinary function, but its use must be confined to certain simple types of expression for which it is obvious that no inconsistency can arise.

The most important property of $\delta(x)$ is exemplified by the following equation,

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0), \tag{3}$$

where $f(x)$ is any continuous function of x. We can easily see the validity of this equation from the above picture of $\delta(x)$. The left-hand side of (3) can depend only on the values of $f(x)$ very close to the origin, so that we may replace $f(x)$ by its value at the origin, $f(0)$, without essential error. Equation (3) then follows from the first of equations (2). By making a change of origin in (3), we can deduce the formula

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a), \tag{4}$$

where a is any real number. Thus *the process of multiplying a function of x by $\delta(x-a)$ and integrating over all x is equivalent to the process of substituting a for x.* This general result holds also if the function of x is not a numerical one, but is a vector or linear operator depending on x.

The range of integration in (3) and (4) need not be from $-\infty$ to ∞, but may be over any domain surrounding the critical point at which the δ function does not vanish. In future the limits of integration will usually be omitted in such equations, it being understood that the domain of integration is a suitable one.

Equations (3) and (4) show that, although an improper function does not itself have a well-defined value, when it occurs as a factor in an integrand the integral has a well-defined value. In quantum theory, whenever an improper function appears, it will be something which is to be used ultimately in an integrand. Therefore it should be possible to rewrite the theory in a form in which the improper functions appear all through only in integrands. One could then eliminate the improper functions altogether. The use of improper functions thus does not involve any lack of rigour in the theory, but is merely a convenient notation, enabling us to express in a concise form certain relations which we could, if necessary, rewrite in a form not involving improper functions, but only in a cumbersome way which would tend to obscure the argument.

An alternative way of defining the δ function is as the differential coefficient $\epsilon'(x)$ of the function $\epsilon(x)$ given by

$$\left.\begin{aligned} \epsilon(x) &= 0 \quad (x < 0) \\ &= 1 \quad (x > 0). \end{aligned}\right\} \tag{5}$$

We may verify that this is equivalent to the previous definition by substituting $\epsilon'(x)$ for $\delta(x)$ in the left-hand side of (3) and integrating by parts. We find, for g_1 and g_2 two positive numbers,

$$\int_{-g_2}^{g_1} f(x)\epsilon'(x)\,dx = [f(x)\epsilon(x)]_{-g_2}^{g_1} - \int_{-g_2}^{g_1} f'(x)\epsilon(x)\,dx$$

$$= f(g_1) - \int_{0}^{g_1} f'(x)\,dx$$

$$= f(0),$$

in agreement with (3). The δ function appears whenever one differentiates a discontinuous function.

There are a number of elementary equations which one can write down about δ functions. These equations are essentially rules of manipulation for algebraic work involving δ functions. The meaning of any of these equations is that its two sides give equivalent results as factors in an integrand.

Examples of such equations are

$$\delta(-x) = \delta(x) \tag{6}$$

$$x\,\delta(x) = 0, \tag{7}$$

$$\delta(ax) = a^{-1}\delta(x) \quad (a > 0), \tag{8}$$

$$\delta(x^2-a^2) = \tfrac{1}{2}a^{-1}\{\delta(x-a)+\delta(x+a)\} \quad (a > 0), \tag{9}$$

$$\int \delta(a-x)\,dx\,\delta(x-b) = \delta(a-b), \tag{10}$$

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{11}$$

Equation (6), which merely states that $\delta(x)$ is an even function of its variable x is trivial. To verify (7) take any continuous function of x, $f(x)$. Then

$$\int f(x)x\,\delta(x)\,dx = 0,$$

from (3). Thus $x\,\delta(x)$ as a factor in an integrand is equivalent to zero, which is just the meaning of (7). (8) and (9) may be verified by similar elementary arguments. To verify (10) take any continuous function of a, $f(a)$. Then

$$\int f(a)\,da \int \delta(a-x)\,dx\,\delta(x-b) = \int \delta(x-b)\,dx \int f(a)\,da\,\delta(a-x)$$

$$= \int \delta(x-b)\,dx\,f(x) = \int f(a)\,da\,\delta(a-b).$$

Thus the two sides of (10) are equivalent as factors in an integrand with a as variable of integration. It may be shown in the same way that they are equivalent also as factors in an integrand with b as variable of integration, so that equation (10) is justified from either of these points of view. Equation (11) is also easily justified, with the help of (4), from two points of view.

Equation (10) would be given by an application of (4) with $f(x) = \delta(x-b)$. We have here an illustration of the fact that we may often use an improper function as though it were an ordinary continuous function, without getting a wrong result.

Equation (7) shows that, whenever one divides both sides of an equation by a variable x which can take on the value zero, one should add on to one side an arbitrary multiple of $\delta(x)$, i.e. from an equation

$$A = B \tag{12}$$

one cannot infer

$$A/x = B/x,$$

but only

$$A/x = B/x+c\,\delta(x), \tag{13}$$

where c is unknown.

As an illustration of work with the δ function, we may consider the differentiation of $\log x$. The usual formula

$$\frac{d}{dx}\log x = \frac{1}{x} \tag{14}$$

requires examination for the neighbourhood of $x = 0$. In order to make the reciprocal function $1/x$ well defined in the neighbourhood of $x = 0$ (in the sense of an improper function) we must impose on it an extra condition, such as that its integral from $-\epsilon$ to ϵ vanishes. With this extra condition, the integral of the right-hand side of (14) from $-\epsilon$ to ϵ vanishes, while that of the left-hand side of (14) equals $\log(-1)$, so that (14) is not a correct equation. To correct it, we must remember that, taking principal values, $\log x$ has a pure imaginary term $i\pi$ for negative values of x. As x passes through the value zero this pure imaginary term vanishes discontinuously. The differentiation of this pure imaginary term gives us the result $-i\pi\,\delta(x)$, so that (14) should read

$$\frac{d}{dx}\log x = \frac{1}{x}-i\pi\,\delta(x). \tag{15}$$

The particular combination of reciprocal function and δ function appearing in (15) plays an important part in the quantum theory of collision processes (see § 50).