will be a solution of the Helmholtz equation $-\triangle f = k^2 f$ if $R_{k,\ell}$ is a linear combination of the spherical Bessel functions j_ℓ (8.77) and n_ℓ (8.79)

$$R_{k,\ell}(r) = a_{k,\ell} j_\ell(kr) + b_{k,\ell} n_\ell(kr) \tag{8.89}$$

if $\Phi_m = e^{im\phi}$, and if $\Theta_{\ell,m}$ satisfies the associated Legendre equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta_{\ell,m}}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]\Theta_{\ell,m} = 0. \tag{8.90}$$

8.12 The Associated Legendre Functions/Polynomials

The associated Legendre functions $P_\ell^m(x) \equiv P_{\ell,m}(x)$ are polynomials in $\sin\theta$ and $\cos\theta$. They arise as solutions of the separated θ equation (8.90)

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_{\ell,m}}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_{\ell,m} = 0 \tag{8.91}$$

of the laplacian in spherical coordinates. In terms of $x = \cos\theta$, this self-adjoint ordinary differential equation is

$$\left[(1-x^2)P'_{\ell,m}(x)\right]' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P_{\ell,m}(x) = 0. \tag{8.92}$$

The associated Legendre function $P_{\ell,m}(x)$ is simply related to the mth derivative $P_\ell^{(m)}(x)$

$$P_{\ell,m}(x) \equiv (1-x^2)^{m/2}\, P_\ell^{(m)}(x). \tag{8.93}$$

To see why this function satisfies the differential equation (8.92), we differentiate

$$P_\ell^{(m)}(x) = (1-x^2)^{-m/2} P_{\ell,m}(x) \tag{8.94}$$

twice getting

$$P_\ell^{(m+1)} = (1-x^2)^{-m/2}\left(P'_{\ell,m} + \frac{mxP_{\ell,m}}{1-x^2}\right) \tag{8.95}$$

and

$$P_\ell^{(m+2)} = (1-x^2)^{-m/2}\left[P''_{\ell,m} + \frac{2mxP'_{\ell,m}}{1-x^2} + \frac{mP_{\ell,m}}{1-x^2} + \frac{m(m+2)x^2P_{\ell,m}}{(1-x^2)^2}\right]. \tag{8.96}$$

Next we use Leibniz's rule (4.46) to differentiate Legendre's equation (8.28)

$$\left[(1-x^2)\, P_\ell'\right]' + \ell(\ell+1)\, P_\ell = 0 \tag{8.97}$$

m times, obtaining

$$(1-x^2)P_\ell^{(m+2)} - 2x(m+1)P_\ell^{(m+1)} + (\ell-m)(\ell+m+1)P_\ell^{(m)} = 0. \tag{8.98}$$

Now we put the formulas for the three derivatives (8.94–8.96) into this equation (8.98) and find that the $P_{\ell,m}(x)$ as defined (8.93) obey the desired differential equation (8.92).

Thus the associated Legendre functions are

$$P_{\ell,m}(x) = (1-x^2)^{m/2}\, P_\ell^{(m)}(x) = (1-x^2)^{m/2}\, \frac{d^m}{dx^m} P_\ell(x) \tag{8.99}$$

They are simple polynomials in $x = \cos\theta$ and $\sqrt{1-x^2} = \sin\theta$

$$P_{\ell,m}(\cos\theta) = \sin^m\theta\, \frac{d^m}{d(\cos\theta)^m}\, P_\ell(\cos\theta). \tag{8.100}$$

It follows from Rodrigues's formula (8.8) for the Legendre polynomial $P_\ell(x)$ that $P_{\ell,m}(x)$ is given by the similar formula

$$P_{\ell,m}(x) = \frac{(1-x^2)^{m/2}}{2^\ell \ell!}\, \frac{d^{\ell+m}}{dx^{\ell+m}}\, (x^2-1)^\ell \tag{8.101}$$

which tells us that under parity $P_\ell^m(x)$ changes by $(-1)^{\ell+m}$

$$P_{\ell,m}(-x) = (-1)^{\ell+m}\, P_{\ell,m}(x). \tag{8.102}$$

Rodrigues's formula (8.101) for the associated Legendre function makes sense as long as $\ell + m \geq 0$. This last condition is the requirement in quantum mechanics that m not be less than $-\ell$. And if m exceeds ℓ, then $P_{\ell,m}(x)$ is given by more than 2ℓ derivatives of a polynomial of degree 2ℓ; so $P_{\ell,m}(x) = 0$ if $m > \ell$. This last condition is the requirement in quantum mechanics that m not be greater than ℓ. So we have

$$-\ell \leq m \leq \ell. \tag{8.103}$$

One may show that

$$P_{\ell,-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!}\, P_{\ell,m}(x). \tag{8.104}$$

In fact, since m occurs only as m^2 in the ordinary differential equation (8.92), $P_{\ell,-m}(x)$ must be proportional to $P_{\ell,m}(x)$.

Under reflections, the parity of $P_{\ell,m}$ is $(-1)^{\ell+m}$, that is,

$$P_{\ell,m}(-x) = (-1)^{\ell+m} P_{\ell,m}(x). \tag{8.105}$$

If $m \neq 0$, then $P_{\ell,m}(x)$ has a power of $\sqrt{1-x^2}$ in it, so

$$P_{\ell,m}(\pm 1) = 0 \quad \text{for} \quad m \neq 0. \tag{8.106}$$

We may consider either $\ell(\ell+1)$ or m^2 as the eigenvalue in the ODE (8.92)

$$\left[(1-x^2)P'_{\ell,m}(x)\right]' + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right] P_{\ell,m}(x) = 0. \tag{8.107}$$

If $\ell(\ell+1)$ is the eigenvalue, then the weight function is unity, and since this ODE is self adjoint on the interval $[-1,1]$ (at the ends of which $p(x) = (1-x^2) = 0$), the eigenfunctions $P_{\ell,m}(x)$ and $P_{\ell',m}(x)$ must be orthogonal on that interval when $\ell \neq \ell'$. The full integral formula is

$$\int_{-1}^{1} P_{\ell,m}(x)\, P_{\ell',m}(x)\, dx = \frac{2}{2\ell+1}\, \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{\ell,\ell'}. \tag{8.108}$$

If m^2 for fixed ℓ is the eigenvalue, then the weight function is $1/(1-x^2)$, and the eigenfunctions $P_{\ell,m}(x)$ and $P_{\ell,m'}(x)$ must be orthogonal on $[-1,1]$ when $m \neq m'$. The full formula is

$$\int_{-1}^{1} P_{\ell,m}(x)\, P_{\ell,m'}(x)\, \frac{dx}{1-x^2} = \frac{(\ell+m)!}{m(\ell-m)!}\, \delta_{m,m'}. \tag{8.109}$$

8.13 Spherical Harmonics

The spherical harmonic $Y_\ell^m(\theta,\phi) \equiv Y_{\ell,m}(\theta,\phi)$ is the product

$$Y_{\ell,m}(\theta,\phi) = \Theta_{\ell,m}(\theta)\, \Phi_m(\phi) \tag{8.110}$$

in which $\Theta_{\ell,m}(\theta)$ is proportional to the associated Legendre function $P_{\ell,m}$

$$\Theta_{\ell,m}(\theta) = (-1)^m \sqrt{\frac{2\ell+1}{2}\, \frac{(\ell-m)!}{(\ell+m)!}}\; P_{\ell,m}(\cos\theta) \tag{8.111}$$

and

$$\Phi_m(\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}. \tag{8.112}$$

The big square-root in the definition (8.111) ensures that

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; Y^*_{\ell,m}(\theta,\phi)\, Y_{\ell',m'}(\theta,\phi) = \delta_{\ell\ell'}\, \delta_{mm'}. \tag{8.113}$$