6.18 In example 6.29, show that the solutions associated with the roots $r = 0$ and $r = 1$ are the same.

6.19 For a hydrogen atom, we set $V(r) = -e^2/4\pi\epsilon_0 r \equiv -q^2/r$ in (6.439) and get $(r^2 R'_{n,\ell})' + \left[(2m/\hbar^2)\left(E_{n,\ell} + Zq^2/r\right) r^2 - \ell(\ell+1)\right] R_{n,\ell} = 0$. So at big r, $R''_{n,\ell} \approx -2mE_{n,\ell} R_{n,\ell}/\hbar^2$ and $R_{n,\ell} \sim \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar)$. At tiny r, $(r^2 R'_{n,\ell})' \approx \ell(\ell+1) R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. Set $R_{n,\ell}(r) = r^\ell \exp(-\sqrt{-2mE_{n,\ell}}\,r/\hbar) P_{n,\ell}(r)$ and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.

6.20 Show that as long as the matrix $\mathcal{Y}_{kj} = y_k^{(\ell_j)}(x_j)$ is nonsingular, the n boundary conditions

$$b_j = y^{(\ell_j)}(x_j) = \sum_{k=1}^{n} c_k \, y_k^{(\ell_j)}(x_j) \tag{6.440}$$

determine the n coefficients c_k of the expansion (6.222) to be

$$C^{\mathsf{T}} = B^{\mathsf{T}} \mathcal{Y}^{-1} \quad \text{or} \quad C_k = \sum_{j=1}^{n} b_j \mathcal{Y}_{jk}^{-1}. \tag{6.441}$$

6.21 Show that if the real and imaginary parts u_1, u_2, v_1, and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, then its complex analog (6.242) also vanishes.

6.22 Show that if the real and imaginary parts u_1, u_2, v_1, and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, and if the differential operator L is real and self adjoint, then (6.238) implies (6.243).

6.23 Show that if D is the set of all twice-differentiable functions $u(x)$ on $[a, b]$ that satisfy Dirichlet's boundary conditions (6.245) and if the function $p(x)$ is continuous and positive on $[a, b]$, then the adjoint set D^* defined as the set of all twice-differentiable functions $v(x)$ that make the boundary term (6.247) vanish for all functions $u \in D$ is D itself.

6.24 Same as exercise (6.23) but for Neumann boundary conditions (6.246).

6.25 Use Bessel's equation (6.307) and the boundary conditions $u(0) = 0$ for $n > 0$ and $u(1) = 0$ to show that the eigenvalues λ are all positive.

6.26 Show that after the change of variables $u(x) = J_n(kx) = J_n(\rho)$, the self-adjoint differential equation (6.307) becomes Bessel's equation (6.308).

6.27 Derive Bessel's inequality (6.378) from the inequality (6.377).

6.28 Repeat example 6.41 using J_1's instead of J_0's. Hint: the *Mathematica* command Do[Print[N[BesselJZero[1, k], 10]], {k, 1, 100, 1}] gives the first 100 zeros $z_{1,k}$ of the Bessel function $J_1(x)$ to 10 significant figures.