6.2 Show that since the Bessel function $J_n(x)$ satisfies Bessel's equation (6.48), the function $P_n(\rho) = J_n(k\rho)$ satisfies (6.47).

6.3 Show that (6.58) implies that $R_{k,\ell}(r) = j_\ell(kr)$ satisfies (6.57).

6.4 Use (6.56, 6.57), and $\Phi''_m = -m^2\Phi_m$ to show in detail that the product $f(r,\theta,\phi) = R_{k,\ell}(r)\,\Theta_{\ell,m}(\theta)\,\Phi_m(\phi)$ satisfies $-\triangle f = k^2 f$.

6.5 Replacing Helmholtz's k^2 by $2m(E - V(r))/\hbar^2$, we get Schrödinger's equation

$$-(\hbar^2/2m)\triangle\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi). \tag{6.438}$$

Let $\psi(r,\theta,\phi) = R_{n,\ell}(r)\Theta_{\ell,m}(\theta)e^{im\phi}$ in which $\Theta_{\ell,m}$ satisfies (6.56) and show that the radial function $R_{n,\ell}$ must obey

$$-\left(r^2 R'_{n,\ell}\right)'/r^2 + \left[\ell(\ell+1)/r^2 + 2mV/\hbar^2\right]R_{n,\ell} = 2mE_{n,\ell}\,R_{n,\ell}/\hbar^2. \tag{6.439}$$

6.6 Use the empty-space Maxwell's equations $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, $\boldsymbol{\nabla}\times\boldsymbol{E} + \dot{\boldsymbol{B}} = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$, and $\boldsymbol{\nabla}\times\boldsymbol{B} - \dot{\boldsymbol{E}}/\boldsymbol{c^2} = 0$ and the formula (6.437) to show that in vacuum $\triangle\boldsymbol{E} = \ddot{\boldsymbol{E}}/c^2$ and $\triangle\boldsymbol{B} = \ddot{\boldsymbol{B}}/c^2$.

6.7 Argue from symmetry and anti-symmetry that $[\gamma^a, \gamma^b]\partial_a\partial_b = 0$ in which the sums over a and b run from 0 to 3.

6.8 Suppose a voltage $V(t) = V\sin(\omega t)$ is applied to a resistor of R (Ω) in series with a capacitor of capacitance C (F). If the current through the circuit at time $t = 0$ is zero, what is the current at time t?

6.9 (a) Is $\left(1 + x^2 + y^2\right)^{-3/2}\left[(1+y^2)y\,dx + (1+x^2)x\,dy\right] = 0$ exact? (b) Find its general integral and solution $y(x)$. Use section 6.11.

6.10 (a) Separate the variables of the ODE $(1+y^2)y\,dx + (1+x^2)x\,dy = 0$. (b) Find its general integral and solution $y(x)$.

6.11 Find the general solution to the differential equation $y' + y/x = c/x$.

6.12 Find the general solution to the differential equation $y' + xy = ce^{-x^2/2}$.

6.13 James Bernoulli studied ODEs of the form $y' + p\,y = q\,y^n$ in which p and q are functions of x. Division by y^n and the substitution $v = y^{1-n}$ gives us the equation $v' + (1-n)p\,v = (1-n)\,q$ which is soluble as shown in section (6.16). Use this method to solve the ODE $y' - y/2x = 5x^2y^5$.

6.14 Integrate the ODE $(xy+1)\,dx + 2x^2(2xy-1)\,dy = 0$. Hint: Use the variable $v(x) = xy(x)$ instead of $y(x)$.

6.15 Show that the points $x = \pm 1$ and ∞ are regular singular points of Legendre's equation (6.181).

6.16 Use the vanishing of the coefficient of every power of x in (6.185) and the notation (6.187) to derive the recurrence relation (6.188).

6.17 In example 6.29, derive the recursion relation for $r = 1$ and discuss the resulting eigenvalue equation.

6.18 In example 6.29, show that the solutions associated with the roots $r = 0$ and $r = 1$ are the same.

6.19 For a hydrogen atom, we set $V(r) = -e^2/4\pi\epsilon_0 r \equiv -q^2/r$ in (6.439) and get $(r^2 R'_{n,\ell})' + \left[(2m/\hbar^2)\left(E_{n,\ell} + Zq^2/r\right) r^2 - \ell(\ell+1)\right] R_{n,\ell} = 0$. So at big r, $R''_{n,\ell} \approx -2mE_{n,\ell} R_{n,\ell}/\hbar^2$ and $R_{n,\ell} \sim \exp(-\sqrt{-2mE_{n,\ell}}\, r/\hbar)$. At tiny r, $(r^2 R'_{n,\ell})' \approx \ell(\ell+1) R_{n,\ell}$ and $R_{n,\ell}(r) \sim r^\ell$. Set $R_{n,\ell}(r) = r^\ell \exp(-\sqrt{-2mE_{n,\ell}}\, r/\hbar) P_{n,\ell}(r)$ and apply the method of Frobenius to find the values of $E_{n,\ell}$ for which $R_{n,\ell}$ is suitably normalizable.

6.20 Show that as long as the matrix $\mathcal{Y}_{kj} = y_k^{(\ell_j)}(x_j)$ is nonsingular, the n boundary conditions

$$b_j = y^{(\ell_j)}(x_j) = \sum_{k=1}^{n} c_k\, y_k^{(\ell_j)}(x_j) \tag{6.440}$$

determine the n coefficients c_k of the expansion (6.222) to be

$$C^{\mathsf{T}} = B^{\mathsf{T}}\, \mathcal{Y}^{-1} \quad \text{or} \quad C_k = \sum_{j=1}^{n} b_j \mathcal{Y}^{-1}_{jk}. \tag{6.441}$$

6.21 Show that if the real and imaginary parts u_1, u_2, v_1, and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, then its complex analog (6.242) also vanishes.

6.22 Show that if the real and imaginary parts u_1, u_2, v_1, and v_2 of ψ and χ satisfy boundary conditions at $x = a$ and $x = b$ that make the boundary term (6.240) vanish, and if the differential operator L is real and self adjoint, then (6.238) implies (6.243).

6.23 Show that if D is the set of all twice-differentiable functions $u(x)$ on $[a, b]$ that satisfy Dirichlet's boundary conditions (6.245) and if the function $p(x)$ is continuous and positive on $[a, b]$, then the adjoint set D^* defined as the set of all twice-differentiable functions $v(x)$ that make the boundary term (6.247) vanish for all functions $u \in D$ is D itself.

6.24 Same as exercise (6.23) but for Neumann boundary conditions (6.246).

6.25 Use Bessel's equation (6.307) and the boundary conditions $u(0) = 0$ for $n > 0$ and $u(1) = 0$ to show that the eigenvalues λ are all positive.

6.26 Show that after the change of variables $u(x) = J_n(kx) = J_n(\rho)$, the self-adjoint differential equation (6.307) becomes Bessel's equation (6.308).

6.27 Derive Bessel's inequality (6.378) from the inequality (6.377).

6.28 Repeat example 6.41 using J_1's instead of J_0's. Hint: the *Mathematica* command Do[Print[N[BesselJZero[1, k], 10]], {k, 1, 100, 1}] gives the first 100 zeros $z_{1,k}$ of the Bessel function $J_1(x)$ to 10 significant figures.