V. The integral of the divergence $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ over the tiny volumes dV of the tiny cubes that make up the volume V is the sum of the surface integrals dS over the faces of these tiny cubes. The integrals over the interior faces cancel leaving just the surface integral over the boundary ∂V of the finite volume V. Thus we arrive at Stokes's theorem

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{v} \, dV = \int_{\partial V} \boldsymbol{v} \cdot \boldsymbol{dS}. \tag{6.32}$$

The laplacian is the divergence (6.29) of the gradient (6.26). So in orthogonal coordinates it is

$$\triangle f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla f} = \frac{1}{h_1 h_2 h_3} \left[\sum_{k=1}^{3} \frac{\partial}{\partial u_k} \left(\frac{h_1 h_2 h_3}{h_k^2} \frac{\partial f}{\partial u_k} \right) \right]. \tag{6.33}$$

Thus in cylindrical coordinates, the laplacian is

$$\triangle f = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 f}{\partial \phi^2} + \rho \frac{\partial^2 f}{\partial z^2} \right] = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{6.34}$$

and in spherical coordinates it is

$$\begin{aligned} \triangle f &= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin\theta \frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{1}{\sin\theta} \frac{\partial f}{\partial \phi} \right) \right] \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}. \end{aligned} \tag{6.35}$$

The area $\boldsymbol{dS}$ of a tiny rectangle dS whose sides are the tiny perpendicular vectors $h_i \hat{\boldsymbol{e}}_i du_i$ and $h_j \hat{\boldsymbol{e}}_j du_j$ (no sum) is their cross-product

$$\boldsymbol{dS} = h_i \hat{\boldsymbol{e}}_i du_i \times h_j \hat{\boldsymbol{e}}_j du_j = \hat{\boldsymbol{e}}_k \, h_i h_j \, du_i du_j \tag{6.36}$$

in which the perpendicular unit vectors $\hat{\boldsymbol{e}}_i$, $\hat{\boldsymbol{e}}_j$, and $\hat{\boldsymbol{e}}_k$ obey the right-hand rule. The dot-product of this area with the **curl** of a vector $\boldsymbol{v}$, which is $(\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot dS = (\boldsymbol{\nabla} \times \boldsymbol{v})_k \, h_i h_j \, du_i du_j$, is the line integral dL of $\boldsymbol{v}$ along the boundary ∂dS of the rectangle

$$(\boldsymbol{\nabla} \times \boldsymbol{v})_k \, h_i h_j du_i du_j = [\partial_i (h_j v_j) - \partial_j (h_i v_i)] \, du_i du_j. \tag{6.37}$$

Thus, the kth component of the curl is

$$(\nabla \times v)_k = \frac{1}{h_i h_j} \left(\frac{\partial (h_j v_j)}{\partial u_i} - \frac{\partial (h_i v_i)}{\partial u_j} \right) \quad \text{(no sum)}. \tag{6.38}$$

In terms of the Levi-Civita symbol ϵ_{ijk}, which is totally antisymmetric with