3

Fourier and Laplace Transforms

The complex exponentials $\exp(i2\pi nx/L)$ are orthonormal and easy to differentiate (and to integrate), but they are periodic with period L. If one wants to represent functions that are not periodic, a better choice is the complex exponentials $\exp(ikx)$, where k is an arbitrary real number. These orthonormal functions are the basis of the Fourier transform. The choice of complex k leads to the transforms of Laplace, Mellin, and Bromwich.

3.1 The Fourier Transform

The interval $[-L/2, L/2]$ is arbitrary in the Fourier series pair (2.37)

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{i2\pi nx/L}}{\sqrt{L}} \quad \text{and} \quad f_n = \int_{-L/2}^{L/2} f(x) \frac{e^{-i2\pi nx/L}}{\sqrt{L}}\, dx. \tag{3.1}$$

What happens when we stretch this interval without limit, letting $L \to \infty$?

We may use the **nearest-integer function** $[y]$ to convert the coefficients f_n into a function of a continuous variable $\hat{f}(y) \equiv f_{[y]}$ such that $\hat{f}(y) = f_n$ when $|y - n| < 1/2$. In terms of this function $\hat{f}(y)$, the Fourier series (3.1) for the function $f(x)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} \int_{n-1/2}^{n+1/2} \hat{f}(y) \frac{e^{i2\pi[y]x/L}}{\sqrt{L}}\, dy = \int_{-\infty}^{\infty} \hat{f}(y) \frac{e^{i2\pi[y]x/L}}{\sqrt{L}}\, dy. \tag{3.2}$$

Since $[y]$ and y differ by no more than 1/2, the absolute value of the difference between $\exp(i\pi[y]x/L)$ and $\exp(i\pi yx/L)$ for fixed x is

$$\left| e^{i2\pi[y]x/L} - e^{i2\pi yx/L} \right| = \left| e^{i2\pi([y]-y)x/L} - 1 \right| \approx \frac{\pi|x|}{L} \tag{3.3}$$

which goes to zero as $L \to \infty$. So in this limit, we may replace $[y]$ by y and