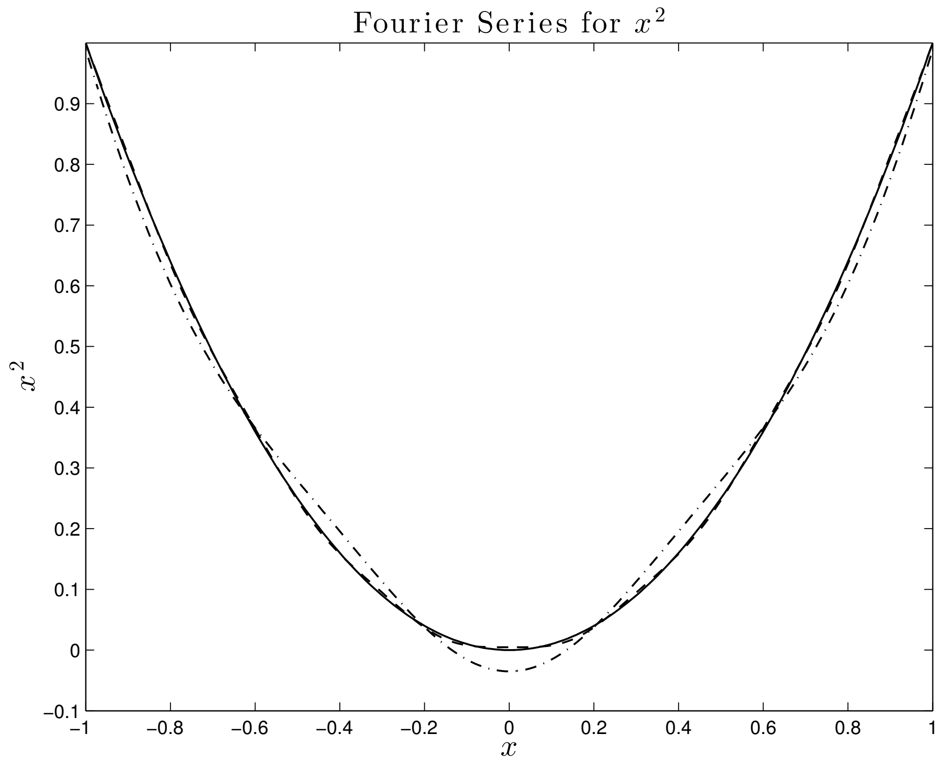

Figure 2.3 The function x^2 (solid) and its Fourier series of 7 terms (dot dash) and 20 terms (dashes). The Fourier series (2.30) for x^2 quickly converges well inside the interval $(-\pi, \pi)$.

for the coefficients a_n and b_n also follow from the orthogonality relations

$$\int_0^{2\pi} \sin mx \, \sin nx \, dx = \begin{cases} \pi & \text{if } n = m \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.24}$$

$$\int_0^{2\pi} \cos mx \, \cos nx \, dx = \begin{cases} \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = m = 0 \\ 0 & \text{otherwise, and} \end{cases} \tag{2.25}$$

$$\int_0^{2\pi} \sin mx \, \cos nx \, dx = 0, \tag{2.26}$$

which hold for integer values of n and m.

What if a function $f(x)$ is not periodic? The Fourier series for an aperiodic function is itself strictly periodic, is sensitive to its interval $(r, r + 2\pi)$ of definition, may differ somewhat from the function near the ends of the interval, and usually differs markedly from it outside the interval.

Example 2.4 (The Fourier Series for x^2) The function x^2 is even and so the integrals (2.23) for its sine Fourier coefficients b_n all vanish. Its cosine coefficients a_n are given by (2.22)

$$a_n = \int_{-\pi}^{\pi} \cos nx\, f(x)\, \frac{dx}{\pi} = \int_{-\pi}^{\pi} \cos nx\, x^2\, \frac{dx}{\pi}. \tag{2.27}$$

Integrating twice by parts, we find for $n \neq 0$

$$a_n = -\frac{2}{n}\int_{-\pi}^{\pi} x \sin nx\, \frac{dx}{\pi} = -\int_{-\pi}^{\pi} \frac{2\cos nx}{\pi n^2} dx + \left[\frac{2x\cos nx}{\pi n^2}\right]_{-\pi}^{\pi} = (-1)^n \frac{4}{n^2} \tag{2.28}$$

and

$$a_0 = \int_{-\pi}^{\pi} x^2\, \frac{dx}{\pi} = \frac{2\pi^2}{3}. \tag{2.29}$$

Equation (2.20) now gives for x^2 the cosine Fourier series

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}. \tag{2.30}$$

This series rapidly converges within the interval $[-1, 1]$ as shown in Fig. 2.3, but not near the endpoints $\pm\pi$. □

Example 2.5 (The Gibbs Overshoot) The function $f(x) = x$ on the interval $[-\pi, \pi]$ is not periodic. So we expect trouble if we represent it as a Fourier series. Since x is an odd function, equation (2.22) tells us that the coefficients a_n all vanish. By (2.23), the b_n's are

$$b_n = \int_{-\pi}^{\pi} \frac{dx}{\pi}\, x \sin nx = 2\,(-1)^{n+1}\,\frac{1}{n}. \tag{2.31}$$

As shown in Fig. 2.4, the series

$$\sum_{n=1}^{\infty} 2\,(-1)^{n+1}\,\frac{1}{n}\,\sin nx \tag{2.32}$$

differs by about 2π from the function $f(x) = x$ for $-3\pi < x < -\pi$ and for $\pi < x < 3\pi$ because the series is periodic while the function x isn't.

Within the interval $(-\pi, \pi)$, the series with 100 terms is very accurate except for $x \gtrsim -\pi$ and $x \lesssim \pi$, where it overshoots by about 9% of the 2π discontinuity, a defect called the **Gibbs phenomenon** or the **Gibbs overshoot** (J. Willard Gibbs 1839–1903. Incidentally Gibbs's father helped defend the Africans of the schooner *Amistad*). Any time we use a Fourier series to represent an aperiodic function, a Gibbs phenomenon will occur near the endpoints of the interval. □

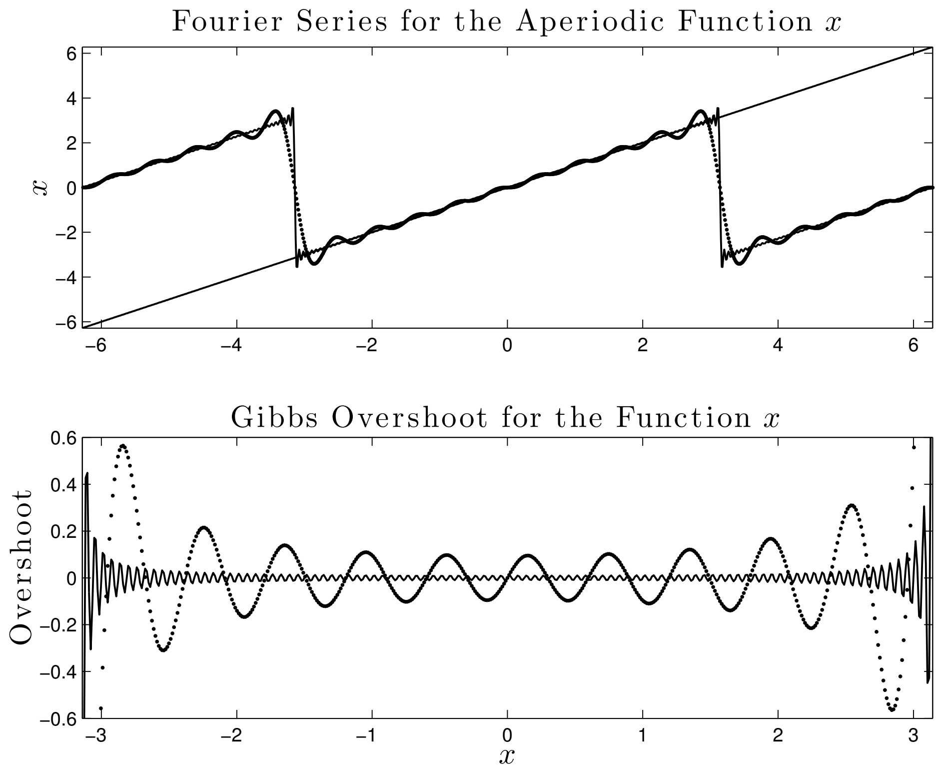

Figure 2.4 (top) The Fourier series (2.32) for the function x (solid line) with 10 terms (dots) and 100 terms (solid curve) for $-2\pi < x < 2\pi$. The Fourier series is periodic, but the function x is not. (bottom) The differences between x and the 10-term (dots) and the 100-term (solid curve) on $(-\pi, \pi)$ exhibit a Gibbs overshoot of about 9% at $x \gtrsim -\pi$ and at $x \lesssim \pi$.

2.5 Stretched Intervals

If the interval of periodicity is of length L instead of 2π, then we may use the phases $\exp(i2\pi nx/\sqrt{L})$ which are orthonormal on the interval $[0, L]$

$$\int_0^L dx \left(\frac{e^{i2\pi nx/L}}{\sqrt{L}}\right)^* \frac{e^{i2\pi mx/L}}{\sqrt{L}} = \delta_{nm}. \tag{2.33}$$

The Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \, \frac{e^{i2\pi nx/L}}{\sqrt{L}} \tag{2.34}$$