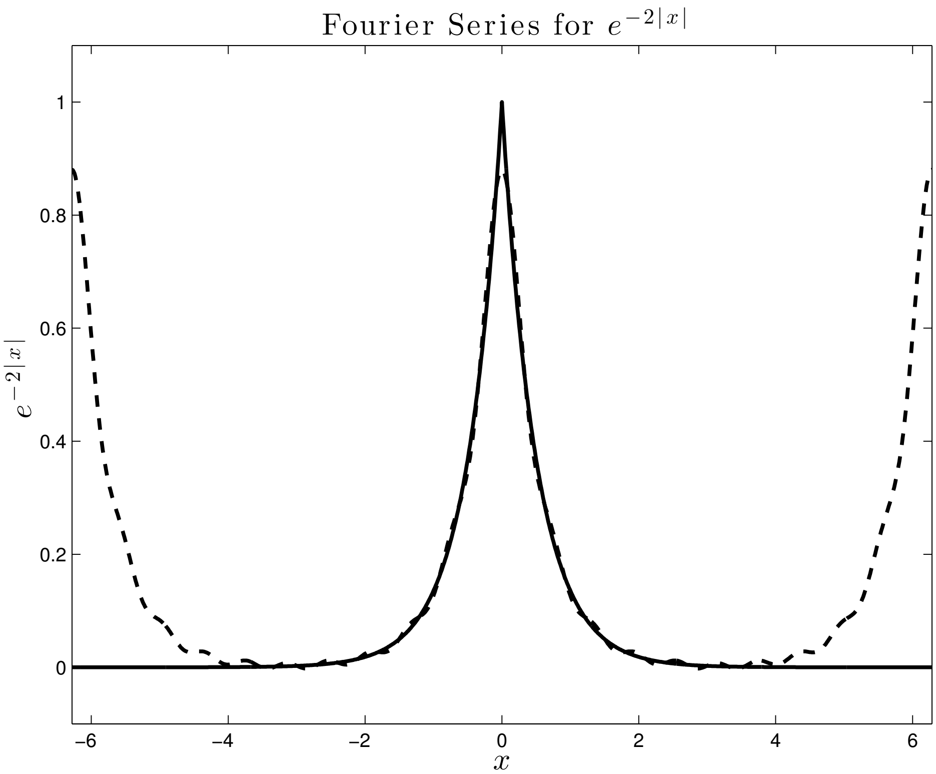

Figure 2.2 The 10-term (dashes) Fourier series (2.17) for the function $\exp(-2|x|)$ on the interval $(-\pi, \pi)$ is plotted from -2π to 2π. All Fourier series are periodic, but the function $\exp(-2|x|)$ (solid) is not.

and (2.3) as

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \, e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi} \int_0^{2\pi} dx \, e^{-inx} \, f(x). \tag{2.12}$$

One also may use the rules

$$f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{and} \quad c_n = \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx. \tag{2.13}$$

Example 2.3 (Fourier Series for $\exp(-m|x|)$) Let's compute the Fourier series for the real function $f(x) = \exp(-m|x|)$ on the interval $(-\pi, \pi)$. Using Eq.(2.10) for the shifted interval and the 2π-placement convention (2.12), we find for the coefficient d_n

$$d_n = \int_{-\pi}^{\pi} \frac{dx}{2\pi}\, e^{-inx}\, e^{-m|x|} \tag{2.14}$$

which we may split into the two pieces

$$d_n = \int_{-\pi}^{0} \frac{dx}{2\pi}\, e^{(m-in)x} + \int_{0}^{\pi} \frac{dx}{2\pi}\, e^{-(m+in)x}. \tag{2.15}$$

After doing the integrals, we find

$$d_n = \frac{1}{\pi}\frac{m}{m^2+n^2}\left[1-(-1)^n\, e^{-\pi m}\right]. \tag{2.16}$$

Here since m is real, $d_n = d_n^*$, but also $d_n = d_{-n}$. So the coefficients d_n satisfy the condition (2.7) that holds when the function $f(x)$ is real, $d_n = d_{-n}^*$. The Fourier series for $\exp(-m|x|)$ with d_n given by (2.16) is

$$\begin{aligned} e^{-m|x|} &= \sum_{n=-\infty}^{\infty} d_n e^{inx} = \sum_{n=-\infty}^{\infty} \frac{1}{\pi}\frac{m}{m^2+n^2}\left[1-(-1)^n\, e^{-\pi m}\right] e^{inx} \\ &= \frac{(1-e^{-\pi m})}{m\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{m}{m^2+n^2}\left[1-(-1)^n\, e^{-\pi m}\right]\cos(nx) \end{aligned} \tag{2.17}$$

In Fig. 2.2, the 10-term (dashes) Fourier series for $m = 2$ is plotted from $x = -2\pi$ to $x = 2\pi$. The function $\exp(-2|x|)$ itself is represented by a solid line. Although it is not periodic, its Fourier series is periodic with period 2π. The 10-term Fourier series represents the function $\exp(-2|x|)$ quite well within the interval $[-\pi, \pi]$.

In what follows, we usually won't bother to use different letters to distinguish between the symmetric (2.2 & 2.3) and asymmetric (2.12 & 2.13) conventions on the placement of the 2π's. □

2.4 Real Fourier Series for Real Functions

The rules (2.1–2.3 and 2.10–2.13) for Fourier series are simple and apply to functions that are continuous and periodic — whether complex or real. If the function $f(x)$ is real, then by (2.7) $d_{-n} = d_n^*$, whence $d_0 = d_0^*$, so d_0 is