Often one uses the notation $|A| = \det A$ to denote a determinant. In this more compact notation, the obvious generalization of the product rule is

$$|ABC \ldots Z| = |A||B| \ldots |Z|. \tag{1.207}$$

The product rule (1.204) implies that $\det\left(A^{-1}\right)$ is $1/\det A$ since

$$1 = \det I = \det\left(AA^{-1}\right) = \det A \det\left(A^{-1}\right). \tag{1.208}$$

Example 1.28 (Derivative of the logarithm of a determinant) We see from our formula (1.195) for $\det A$ that its derivative with respect to any given element A_{ik} is the corresponding cofactor C_{ik}

$$\frac{\partial \det A}{\partial A_{ik}} = C_{ik} \tag{1.209}$$

since the cofactors C_{ij} and C_{jk} for all j are independent of A_{ik}. Thus the derivative of the logarithm of this determinant with respect to any parameter β is

$$\begin{aligned} \frac{\partial \ln \det A}{\partial \beta} &= \frac{1}{\det A} \sum_{ik} \frac{\partial \det A}{\partial A_{ik}} \frac{\partial A_{ik}}{\partial \beta} = \sum_{ik} \frac{C_{ik}}{\det A} \frac{\partial A_{ik}}{\partial \beta} \\ &= \sum_{ik} A^{-1}_{ki} \frac{\partial A_{ik}}{\partial \beta} = \operatorname{Tr}\left(A^{-1} \frac{\partial A}{\partial \beta}\right). \end{aligned} \tag{1.210}$$

□

Incidentally, Gauss, Jordan, and modern mathematicians have developed much faster ways of computing determinants and matrix inverses than those (1.183 & 1.197) due to Laplace. Sage, Octave, Matlab, Maple, Mathematica, and Python use these modern techniques, which also are freely available as programs in C and FORTRAN from www.netlib.org/lapack.

Example 1.29 (Numerical Tricks) Adding multiples of rows to other rows does not change the value of a determinant, and interchanging two rows only changes a determinant by a minus sign. So we can use these operations, which leave determinants invariant, to make a matrix **upper triangular**, a form in which its determinant is just the product of the factors on its diagonal. For instance, to make the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -6 & 3 \\ 4 & 2 & -5 \end{pmatrix} \tag{1.211}$$