and its adjoint $A^\dagger$ is the operator $IA^\dagger I$

$$A^\dagger = \sum_{n,\ell=1}^{N} |n\rangle\langle n|A^\dagger|\ell\rangle\langle\ell| = \sum_{n,\ell=1}^{N} |\ell\rangle\langle n|A|\ell\rangle^*\langle n| = \sum_{n,\ell=1}^{N} |n\rangle\langle\ell|A|n\rangle^*\langle\ell|$$

in which we used the definition (1.147) of the adjoint of an outer product and then interchanged ℓ and n. It follows that $\langle n|A^\dagger|\ell\rangle = \langle\ell|A|n\rangle^*$ so that the matrix $A^\dagger_{n\ell}$ that represents $A^\dagger$ in this basis is

$$A^\dagger_{n\ell} = \langle n|A^\dagger|\ell\rangle = \langle\ell|A|n\rangle^* = A^*_{\ell n} = A^{*\mathsf{T}}_{n\ell} \tag{1.149}$$

in agreement with our definiton (1.28) of the adjoint of a matrix as the transpose of its complex conjugate, $A^\dagger = A^{*\mathsf{T}}$. We also have

$$\langle g|A^\dagger f\rangle = \langle g|A^\dagger|f\rangle = \langle f|A|g\rangle^* = \langle f|Ag\rangle^* = \langle Ag|f\rangle. \tag{1.150}$$

Taking the adjoint of the adjoint is by (1.147)

$$\left[(z\,|n\rangle\langle\ell|)^\dagger\right]^\dagger = [z^*\,|\ell\rangle\langle n|]^\dagger = z\,|n\rangle\langle\ell| \tag{1.151}$$

the same as doing nothing at all. This also follows from the matrix formula (1.149) because both $(A^*)^* = A$ and $(A^\mathsf{T})^\mathsf{T} = A$, and so

$$\left(A^\dagger\right)^\dagger = \left(A^{*\mathsf{T}}\right)^{*\,\mathsf{T}} = A \tag{1.152}$$

the adjoint of the adjoint of a matrix is the original matrix.

Before Dirac, the adjoint $A^\dagger$ of a linear operator A was defined by

$$(g, A^\dagger f) = (A\,g, f) = (f, A\,g)^*. \tag{1.153}$$

This definition also implies that $A^{\dagger\dagger} = A$ since

$$(g, A^{\dagger\dagger} f) = (A^\dagger g, f) = (f, A^\dagger g)^* = (Af, g)^* = (g, Af). \tag{1.154}$$

We also have $(g, Af) = (g, A^{\dagger\dagger} f) = (A^\dagger g, f)$.

1.14 Self-Adjoint or Hermitian Linear Operators

An operator A that is equal to its adjoint $A^\dagger = A$ is **self adjoint** or **hermitian**. In view of (1.149), the matrix elements of a self-adjoint linear operator A satisfy $\langle n|A^\dagger|\ell\rangle = \langle\ell|A|n\rangle^* = \langle n|A|\ell\rangle$ in any orthonormal basis. So a matrix that represents a hermitian operator is equal to the transpose of its complex conjugate

$$A_{n\ell} = \langle n|A|\ell\rangle = \langle n|A^\dagger|\ell\rangle = \langle\ell|A|n\rangle^* = A^{*\mathsf{T}}_{n\ell} = A^\dagger_{n\ell}. \tag{1.155}$$