A differential operator of the form

$$\mathcal{L} = \frac{d}{dx}\left(p(x) \frac{d}{dx} \right) + q(x)$$

is said to be in <u>self-adjoint form</u>.

Why is that good? If $\mathcal{L}$ is self adjoint, then

$$\mathcal{L} u = (p u')' + q u$$

and

$$\int_a^b v^* \mathcal{L} u \, dx = \int_a^b v^* \left[(p u')' + q u \right] dx$$

$$= \int_a^b \left[-p u' v^{*\prime} + u q v^* \right] dx + \left[v^* p u' \right]_a^b$$

$$= \int_a^b \left[u p v^{*\prime\prime} + u q v^* \right] dx - \left[p u v^{*\prime} \right]_a^b + \left[v^* p u' \right]_a^b$$

$$= \int_a^b u \mathcal{L} v^* dx + \left[v^* p u' - v^{*\prime} p u \right]_a^b$$

Suppose now that the boundary terms vanish. Then

$$\int_a^b v^* \mathcal{L} u \, dx = \int_a^b u \mathcal{L} v^* dx = \int_a^b (\mathcal{L} v^*) u \, dx.$$

If in fact p and q are real so that $\mathcal{L}^* = \mathcal{L}$, then

$$\begin{array}{ccccc} \int_a^b v^* \mathcal{L} u \, dx & = & \int_a^b (\mathcal{L} v)^* u \, dx & = & \left(\int_a^b u^* \mathcal{L} v \, dx \right)^* \\ \| & & \| & & \| \\ & & \int_a^b v^* \mathcal{L}^\dagger u \, dx & & \\ \langle v | \mathcal{L} | u \rangle & = & \langle v | \mathcal{L}^\dagger | u \rangle & = & \langle u | \mathcal{L} | v \rangle^* \end{array}$$

which we interpret as $\mathcal{L} = \mathcal{L}^\dagger$, ie, the operator $\mathcal{L}$ is hermitian. So a self-adjoint $\mathcal{L}$ with real p & q is hermitian.

The self-adjoint character of $\mathcal{L}$ is effective only if

$$\left[p(v^* u' - v^{*\prime} u) \right]_a^b = 0 .$$

So one must choose a and b carefully to fit each problem. Usually, people require that all the functions u and v to which $\mathcal{L}$ is applied satisfy the boundary conditions

$$p v^* u' \Big]_a^b = 0 = p u v^{*\prime} \Big]_a^b .$$

Sometimes, one even insists that

$$p v^* u' \Big|_{x=a} = 0 = p v^* u' \Big|_{x=b}$$

$$p u^* v' \Big|_{x=a} = 0 \quad p u^* v' \Big|_{x=b}$$

but these extra conditions are not necessary.

In some cases a and b are taken to be $a = -\infty$ and $b = +\infty$, and it is assumed that

$$0 = p v^* u' \quad \text{etc at} \quad x = \pm\infty.$$

One might imagine that self-adjoint operators are rare. But in fact one can multiply the generic 2d-order differential operator

$$\mathcal{L}_0 u = p_0 u'' + p_1 u' + p_2 u$$

by $$\frac{1}{p_0} e^{\int^x \frac{p_1(t)}{p_0(t)} dt}$$

and get a self-adjoint $\mathcal{L}$:

$$\frac{1}{p_0} e^{\int^x \frac{p_1}{p_0} dt} \mathcal{L}_0 u = e^{\int^x p_1/p_0 dt} u'' + \frac{p_1}{p_0} e^{\int^x p_1/p_0 dt} u'$$

$$+ \frac{p_2}{p_0} e^{\int^x p_1/p_0 dt} u$$

$$= \left\{ \exp\left[\int^x \frac{p_1(t) dt}{p_0(t)} \right] u' \right\}' + \frac{p_2}{p_0} \exp \int^x \frac{p_1}{p_0} dt \; u$$

$$= (p u')' + q u \qquad \text{where}$$

$$p = \exp\left[\int^x \frac{p_1(t)}{p_0(t)} dt \right] \quad \text{and} \quad q = \frac{p_2}{p_0} p.$$

So we always may cast a 2d-order differential operator into self-adjoint form, $\mathcal{L}$. And if p & q are real, then $\mathcal{L}$ is hermitian on functions u and v that satisfy the boundary conditions

$$\left[p \, (v^* u' - v^{*\prime} u) \right]_a^b = 0 .$$

Eigen This and That

Suppose

$$\lambda w u + \mathcal{L} u = (p u')' + q u + \lambda w u = 0 \qquad \text{ie}$$

$$(p(x) \, u'(x))' + q(x) \, u(x) + \lambda \, w(x) \, u(x) = 0,$$

then λ is said to be the <u>Eigenvalue</u> and $w(x)$ is a known weight or density function. Here $w(x) \geq 0$ and $w(x) = 0$ only at isolated points. '<u>Eigen</u>' means 'special' in Deutsch.

Legendre's eq. is a good example. It is

$$0 = \mathcal{L} y = (1 - x^2) y'' - 2 x y' + \ell(\ell+1) y$$

$$0 = ((1 - x^2) y')' + \ell(\ell+1) y$$

As you will show when you do HW problem 8.5.5, series solutions exist for each of the two solutions $k=0$ and $k=1$ of the indicial equation

$$k(k-1)=0,$$

but the resulting series diverge for $x=\pm 1$ (i.e., $\theta = 0$ or π) unless ℓ is an integer. This is why orbital angular momentum is quantized. So the eigenvalue is $\ell(\ell+1)$ for integral ℓ.

The deuteron is a spin-1 bound state of a neutron and a proton with binding energy about 2.2 MeV. It is mostly an S-state with some d-state mixed in. So the spins of the n and p are aligned. If r is the relative distance in the reduced-mass formalism, then

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = E\psi, \qquad V(r) = \begin{cases} V_0<0 & \text{if } r<a \\ 0 & \text{if } r>a \end{cases}$$

For an s-state boils down to this equation

$$u'' + k_1^2 u = 0 \qquad\qquad u'' - k_2^2 u = 0$$

for $u(r) = r\psi(r)$ where $k_1^2 = \frac{2M}{\hbar^2}(E-V_0) > 0$ for $0 \le r \le a$ and

$$k_2^2 = -\frac{2ME}{\hbar^2} > 0$$

for $a \le r$.

So $u(r) = \alpha \sin k_1 r + \beta \cos k_1 r$ for $r < a$,
but $\beta = 0$ to avoid a singularity at $r = 0$
in

$$\psi(r) = \frac{u(r)}{r}.$$

Outside the square well,

$$u = A e^{k_2 r} + B e^{-k_2 r},$$

but $A = 0$ so that ψ can be normalized.

When one matches the two solutions at the boundary, $r = a$, by requiring that

$$u_1(a) = \alpha \sin k_1 a = B e^{-k_2 a} = u_2(a) \quad \text{and}$$

$$u_1'(a) = \alpha k_1 \cos k_1 a = -k_2 B e^{-k_2 a} = u_2'(a),$$

one finds that

$$\tan k_1 a = -\frac{k_1}{k_2} = -\sqrt{\frac{E - V_0}{-E}}, \quad E < 0, \; V_0 < 0.$$

i.e.,

$$\tan\left(\sqrt{\frac{2Ma^2(E - V_0)}{\hbar^2}}\right) = -\sqrt{\frac{E - V_0}{-E}}$$

which has only discrete, quantized solutions for $E < 0$ and $V_0 < 0$, $|V_0| > |E|$.

$\mathcal{L}$ is 2d order. What about first-order Hermitian operators? Some examples are

$$\vec{p} = \frac{\hbar}{i}\vec{\nabla} \quad \text{or in one dimension} \quad p = \frac{\hbar}{i}\frac{d}{dx}.$$

In this case

$$\int_a^b dx\, v^* p u = \int_a^b dx\, v^* \frac{\hbar}{i} u' = -\int_a^b dx\, u \frac{\hbar}{i} v^{*\prime} + \frac{\hbar}{i}\left[v^* u\right]_a^b$$

$$= \int_a^b dx\, u \left(\frac{\hbar}{i} v'\right)^* + \frac{\hbar}{i}\left[v^* u\right]_a^b$$

So if

$$0 = \left[v^* u\right]_a^b = v^*(b)\,u(b) - v^*(a)\,u(a),$$

then

$$\int_a^b dx\, v^* p u = \left(\int_a^b dx\, u^* p v\right)^* = \int_a^b dx\, (p^\dagger v)^* u$$

$$= \int_a^b dx\, (p v)^* u \qquad \text{so} \quad p = p^\dagger.$$

The trick is in the i, and in the boundary conditions

$$0 = \left[v^* u\right]_a^b$$

which often are satisfied when $a \to -\infty$, $b \to +\infty$, and both u and v are normalized.

Because of operators like $\vec{p}$, we generalize the notion of self-adjoint operators to hermitian operators: those for which

$$\int_a^b dx\, v^* \mathcal{L} u = \int_a^b dx\, (\mathcal{L} v)^* u$$

as long as u and v satisfy suitable boundary conditions at $x = a$ & b. We say $\mathcal{L} = \mathcal{L}^\dagger$.

Suppose $\mathcal{L} = \mathcal{L}^\dagger$, i.e., that $\mathcal{L}$ is hermitian, and that

$$\mathcal{L} u_i + \lambda_i w u_i = 0 \qquad i = 1, 2, \dots$$

Then also

$$\mathcal{L} u_j + \lambda_j w u_j = 0$$

so that

$$(\mathcal{L} u_j)^* + \lambda_j^* w u_j^* = 0 .$$

Note we take $w(x) = w^*(x)$ to be real. Then both

$$u_j^* \mathcal{L} u_i + \lambda_i w u_j^* u_i = 0$$

and

$$u_i (\mathcal{L} u_j)^* + \lambda_j^* w u_j^* u_i = 0$$

so that

$$\underbrace{\int_a^b dx \left[u_j^* \mathcal{L} u_i - u_i (\mathcal{L} u_j)^* \right]}_{=\,0} = (\lambda_j^* - \lambda_i) \int_a^b dx\, w(x)\, u_j^*(x)\, u_i(x) .$$

0 since $\mathcal{L} = \mathcal{L}^\dagger$, as long as the u_i's satisfy the appropriate boundary conditions.

So

$$(\lambda_j^* - \lambda_i) \int_a^b dx \; w(x) \, u_j^*(x) \, u_i(x) = 0 .$$

Set $i = j$. Then

$$(\lambda_i^* - \lambda_i) \int_a^b dx \; w(x) |u_i(x)|^2 = 0 .$$

Since by assumption $w(x) > 0$ except at isolated points, it follows that

$$\lambda_i^* = \lambda_i .$$

The eigenvalues of hermitian operators are <u>real</u>.

So the top equation reads

$$(\lambda_j - \lambda_i) \int_a^b dx \; w(x) \, u_j^*(x) \, u_i(x) = 0 .$$

Thus the eigenfunctions $u_i(x)$ and $u_j(x)$ of different, unequal eigenvalues $\lambda_j \neq \lambda_i$ must be orthogonal.

When two or more eigenfunctions do have the same eigenvalue, they are called <u>degenerate</u>. In the H atom, for instance, states with the same principal quantum number, n, are degenerate to lowest order in the non-relativistic theory.

Suppose several $u_i(x)$ all have the same λ

$$\mathcal{L} u_i + \lambda w u_i = 0 \qquad \text{for } i = 1, 2, \ldots N.$$

Then any linear combination of the u_i's also will satisfy

$$\mathcal{L}\left(\sum_{i=1}^{N} c_i u_i\right) + \lambda w \left(\sum_{i=1}^{N} c_i u_i\right) = 0$$

because $\mathcal{L}$ is a linear differential operator. So we may choose the c_i's so as to make mutually orthogonal linear combinations

$$\psi_i = \sum_{j=1}^{N} c_{ij} u_j .$$

We may even make them orthonormal

$$\phi_i = \frac{\psi_i}{\left[\int_a^b dx\, |\psi_i|^2 w\right]^{1/2}} .$$

The Gram-Schmidt way:

Set $\quad \psi_1(x) = u_1(x)$

$$\phi_1(x) = \frac{\psi_1(x)}{\left[\int_a^b |\psi_1(x)|^2 w(x)\, dx\right]^{1/2}}$$

For $n=2$, we set

$$\psi_2(x) = u_2(x) + a_{21}\,\phi_1(x).$$

We want

$$0 = \int_a^b dx\, \psi_2(x)\,\phi_1^*(x)\,w(x)$$

$$= \int_a^b dx\, u_2(x)\,\phi_1^*(x)\,w(x) + a_{21}^* \int_a^b dx\, |\phi_1(x)|^2 w(x)$$

So

$$a_{21} = -\int_a^b dx\, u_2(x)\,\phi_1^*(x)\,w(x).$$

Then

$$\phi_2(x) = \frac{\psi_2(x)}{\left[\int_a^b dx\, |\psi_2(x)|^2 w(x)\right]^{1/2}},$$

Suppose now that $\phi_1, \phi_2, \dots \phi_i$ are all orthonormal.
We set

$$\psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^{i} a_{i+1,j}\,\phi_j(x).$$ We set

$$0 = \int_a^b dx\, \psi_{i+1}^*(x)\,\phi_j^*(x)\,w(x)$$

$$0 = \int_a^b dx\, u_{i+1}(x)\,\phi_j^*(x)\,w(x) + a_{i+1,j}^*$$ so

$$a_{i+1,j} = -\int_a^b dx\, u_{i+1}(x)\,\phi_j^*(x)\,w(x).$$

Finally

$$\phi_{i+1}(x) = \frac{\psi_{i+1}(x)}{\left[\int_a^b dx\, |\psi_{i+1}(x)|^2 w(x)\right]^{1/2}}.$$

So to gild the lilly, we write

$$\psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^{i} a_{i+1,j}\, \phi_j(x)$$

$$= u_{i+1}(x) - \sum_{j=1}^{i} \int_a^b dx'\, u_{i+1}(x')\, \phi_j^*(x')\, w(x')\, \phi_j(x)$$

If we set

$$P_j u_i(x) = \left[\int_a^b dt\, u_i(t)\, \phi_j^*(t) w(t)\right] \phi_j(x), \text{ then}$$

$$\psi_{i+1} = \left\{1 - \sum_{j=1}^{i} P_j\right\} u_{i+1}(x).$$

Example 9.3.1 Say $u_n(x) = x^n$ $n = 0, 1, 2, \dots$ and the interval is $-1 \le x \le 1$ and $w(x) = 1$. Now $u_0 = 1$, so $\psi_0 = 1$, so $\phi_0 = \frac{1}{\sqrt{2}}$.

$\psi_1 = x + a_{10} \frac{1}{\sqrt{2}}$ and so $a_{10} = -\int_{-1}^{1} \frac{x}{\sqrt{2}}\, dx = 0$

and $\phi_1(x) = \sqrt{\frac{3}{2}}\, x$.

Next

$$\psi_2 = x^2 + a_{20}\frac{1}{\sqrt{2}} + a_{21}\sqrt{\frac{3}{2}}\,x \qquad \text{and}$$

$$a_{20} = -\int_{-1}^{1}\frac{x^2\,dx}{\sqrt{2}} = -\frac{\sqrt{2}}{3}$$

$$a_{21} = -\int_{-1}^{1}\sqrt{\frac{3}{2}}\,x^3\,dx = 0 \qquad \text{and so}$$

$$\psi_2 = x^2 - \frac{1}{3} \quad \text{and} \quad \phi_2(x) = \sqrt{\frac{5}{2}}\,\frac{1}{2}\,(3x^2-1).$$

Eventually

$$\phi_3(x) = \sqrt{\frac{7}{2}}\cdot\frac{1}{2}\,(5x^3 - 3x).$$

It turns out that these are the Legendre polynomials

$$\phi_n(x) = \sqrt{\frac{2n+1}{2}}\;P_n(x)$$

apart from factors that reflect different normalization conditions.

Completeness

~~If~~ If any function $f(x)$ in a certain space of functions S can be represented as

$$f(x) = \sum_{n=0}^{\infty} a_n\,\phi_n(x)$$

in the sense that

$$0 = \lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \right|^2 w(x)\,dx ,$$

then the set of functions $\phi_n(x)$ is said to span that space S or to be complete in S. The coefficients a_n are

$$\int_a^b f(x)\, \phi_n^*(x)\, w(x)\,dx = \int_a^b \sum_{j=0}^{\infty} a_j\, \phi_j(x)\, \phi_n^*(x) w(x)\,dx$$

$$= \sum_{j=0}^{\infty} a_j\, \delta_{jn} = a_n \quad \text{so}$$

$$a_n = \int_a^b f(x)\, \phi_n^*(x)\, w(x)\,dx .$$

But when $\mathcal{L}u = (pu')' + qu$ with p and q real

$$0 = \mathcal{L}u + \lambda w u \quad \text{and} \quad w \geq 0, \text{ then}$$

$$0 = (\mathcal{L}u)^* + \lambda w u^* = \mathcal{L}u^* + \lambda w u^* .$$

So u^* is also an eigenfunction and so we may replace u by the real function $\frac{u+u^*}{2}$ suitably normalized.

Typical spaces S are the space $\mathcal{L}_2$ of all square-integrable functions and the space $\mathcal{P}$ of all piece-wise continuous functions. The proof that the eigenfunctions of any class of hermitian operators are complete in $\mathcal{L}_2$ or $\mathcal{P}$ is beyond the scope of this course.

But in the case of the sets of orthogonal polynomials — the Legendre polynomials and others listed in Table 9.3 — we can say more. These polynomials are equivalent to the powers of x, x^n for $n \geq 0$. So we have half of a Laurent series or a whole power series.

If the ϕ_n's are complete for a space S that includes the function f, then

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{where} \quad a_n = \int_a^b f(x) w(x) \phi_n^*(x)\, dx \quad \text{so}$$

$$f(x) = \sum_{n=0}^{\infty} \int_a^b f(y) w(y) \phi_n^*(y)\, dy\, \phi_n(x)$$

$$= \int_a^b dy\, f(y) \left[w(y) \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(y) \right] = \int_a^b dy\, f(y)\, \delta(x-y)$$

So

$$\delta(x-y) = (w(x) w(y))^{\frac{1}{2}} \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(y).$$

Completeness leads to a formula for $\delta(x-y)$.

Bessel's inequality is

$$0 \le \int_a^b dx\, w(x)\, \left| f(x) - \sum_{i=0}^{\infty} a_i \phi_i \right|^2$$

$$0 \le \int_a^b dx\, w(x)\, |f(x)|^2 - \int_a^b dx\, w(x)\, f(x) \sum a_i^* \phi_i^*$$

$$- \int_a^b dx\, w(x)\, f^*(x) \sum a_i \phi_i + \int_a^b dx\, w \sum_{ij} a_i^* a_j \phi_i^* \phi_j \quad \text{or}$$

$$0 \le \int_a^b dx\, w(x) |f(x)|^2 - \sum |a_i|^2 - \sum |a_i|^2 + \sum |a_i|^2$$

or

$$\int_a^b dx\, w(x)\, |f(x)|^2 \ge \sum_{i=0}^{\infty} |a_i|^2 .$$

In many cases, the absolute-value signs are superfluous.

Schwarz's inequality

Let $\psi = f + \lambda g$, so that

$$0 \le \int_a^b |\psi|^2 dx = \int_a^b |f|^2 dx + \lambda \int_a^b f^* g\, dx + \lambda^* \int_a^b g^* f\, dx + |\lambda|^2 \int_a^b |g|^2 dx$$

Treating λ and λ^* as independent variables, we get

$$0 = \int_a^b f^* g\, dx + \lambda^* \int_a^b |g|^2 dx \qquad \text{and}$$

$$0 = \int_a^b g^* f \, dx + \lambda \int_a^b |g|^2 \, dx$$

With these values of λ and λ^*, we get

$$0 \le \int_a^b |\psi|^2 dx = \int_a^b |f|^2 dx - \left(\int_a^b g^* f \, dx \Big/ \int_a^b |g|^2 dx \right) \int_a^b f^* g \, dx$$

$$- \int_a^b g^* f \, dx \left(\int_a^b g f^* dx \Big/ \int_a^b |g|^2 dx \right) + \frac{\int_a^b f g^* dx \int_a^b g f^* dx}{\int_a^b |g|^2 dx}$$

so that

$$\int_a^b |f|^2 dx \int_a^b |g|^2 dx \ge \left| \int_a^b g^* f \, dx \right|^2 .$$

The vector analogy is

$$|\vec{a}^* \cdot \vec{b}|^2 \le \vec{a}^* \cdot \vec{a} \;\; \vec{b}^* \cdot \vec{b}$$

or in Dirac notation

$$|\langle \phi | \psi \rangle|^2 \le \langle \phi | \phi \rangle \langle \psi | \psi \rangle .$$

So when ϕ & ψ are normalized, the probability of finding ψ as ϕ is ≤ 1

$$P(\phi, \psi) \le 1.$$

The review on pages 609–613 is worth reading.

Suppose $\mathcal{L} = \mathcal{L}^\dagger$ is a hermitian operator with eigenfunctions ϕ_n and eigenvalues λ_n

$$\mathcal{L}\phi_n + \lambda_n \phi_n = 0.$$

So $w(x) = 1$ here. Build the Green's function

$$G(x,y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}.$$

See

$$(\mathcal{L} + \lambda) G(x,y) = \sum_{n=0}^{\infty} \frac{(\mathcal{L}+\lambda)\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}$$

$$= \sum_{n=0}^{\infty} (\lambda - \lambda_n) \frac{\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}$$

$$= - \sum_{n=0}^{\infty} \phi_n(x)\phi_n^*(y) = -\delta(x-y).$$

So

$$(\mathcal{L} + \lambda) G(x,y) = -\delta(x-y).$$

So if we have $\mathcal{L}\psi + \lambda\psi = -\rho$, then we try

$$\psi(x) = \int d^3y \, G(x,y)\rho(y) \qquad \text{so that}$$

$$(\mathcal{L}+\lambda)\psi(x) = \int d^3x \, (\mathcal{L}+\lambda) G(x,y)\rho(y) = -\int d^3y \, \delta(x-y)\rho(y) = -\rho(x).$$

A more explicit treatment is available in one dimension: We will take $\mathcal{L}$ to be real and self adjoint

$$\mathcal{L}u = (pu')' + qu$$ and we want to solve

$$\mathcal{L}y(x) + f(x) = 0.$$

Try
$$G(x,y) = -\frac{1}{A}\begin{cases} u(x)v(y) & a \le x < y \\ u(y)v(x) & y < x \le b \end{cases}$$

here A is a constant
and $\mathcal{L}u = \mathcal{L}v = 0$ and u and v respectively satisfy suitable boundary conditions at $x=a$ and $x=b$:

$$u(a)u'(a) = 0 \quad \text{or} \quad \alpha u(a) + \beta u'(a) = 0$$
$$v(b)v'(b) = 0 \quad \text{or} \quad \alpha v(b) + \beta v'(b) = 0.$$

Set
$$y(x) = \int_a^b dy\, G(x,y)\, f(y)$$

$$= -\frac{1}{A}\int_a^x dy\, v(x)u(y)f(y) - \frac{1}{A}\int_x^b dy\, u(x)v(y)f(y).$$

So

$$y'(x) = -\frac{v'(x)}{A}\int_a^x dy\, u(y)f(y) - \frac{u'(x)}{A}\int_x^b dy\, v(y)f(y)$$

$$\left(-\frac{1}{A}v(x)u(x)f(x) + \frac{1}{A}u(x)v(x)f(x) = 0\right)$$

$$y''(x) = -\frac{v''(x)}{A}\int_a^x dy\, u(y)\, f(y) \;-\; \frac{u''(x)}{A}\int_x^b dy\, v(y)\, f(y)$$

$$-\frac{1}{A}\left[u(x)\,v'(x) - u'(x)\,v(x)\right] f(x).$$

Wronsky strikes again! Note that since u & v satisfy

$$0 = \mathcal{L}u = (pu')' + qu = \mathcal{L}v = (pv')' + qv = 0,$$

the Wronskian

$$W = uv' - u'v \qquad \text{satisfies}$$

$$W' = uv'' - u''v$$

Now
$0 = \mathcal{L}u$ implies $pu'' = -p'u' - qu$ so $u'' = -\frac{p'}{p}u' - \frac{q}{p}u$
and $0 = \mathcal{L}v$ implies $v'' = -\frac{p'}{p}v' - \frac{q}{p}v$. So

$$W' = u\left(-\frac{p'v'}{p} - \frac{qv}{p}\right) - \left(-\frac{p'u'}{p} - \frac{q}{p}u\right)v$$

$$= -\frac{p'}{p}\,(uv' - vu') = -\frac{p'}{p}\,W \qquad \text{So}$$

$$\frac{W'}{W} = -\frac{p'}{p} \qquad (\log W)' = -(\log p)'$$

So

$$\log W = -\log p + c$$

$$W = \frac{A}{p}. \qquad \text{So} \qquad \frac{W}{A} = \frac{1}{p}.$$

So

$$y''(x) = -\frac{v''(x)}{A}\int_a^x u(y)f(y)dy - \frac{u''(x)}{A}\int_x^b v(y)f(y)dy - \frac{f(x)}{p(x)}$$

So

$$qy + py'' + p'y' = \mathcal{L}y = \frac{-qv - pv'' - p'v'}{A}\int_a^x u(y)f(y)dy - \frac{(-qu - pu'' - p'u')}{A}\int_x^b v(y)f(y)dy$$

$$- f(x) \quad \text{or}$$

$$\mathcal{L}y = -\frac{\mathcal{L}v}{A}\int_a^x u(y)f(y)dy - \frac{\mathcal{L}u}{A}\int_x^b v(y)f(y)dy - f(x)$$

But $\mathcal{L}u = \mathcal{L}v = 0$. So

$$\mathcal{L}y + f = 0 \quad \text{or} \quad \mathcal{L}y(x) + f(x) = 0.$$

Note that $y(x)$ satisfies the same boundary conditions at $x = a, b$ as u and v:

$$y(a) = -\frac{1}{A}\int_a^b dy\, u(a)v(y)f(y) = \left(-\frac{1}{A}\int_a^b dy\, v(y)f(y)\right)u(a)$$

$$y'(a) = \left(-\frac{1}{A}\int_a^b dy\, v(y)f(y)\right)u'(a)$$

$$y(b) = \left(-\frac{1}{A}\int_a^b dy\, u(y)f(y)\right)v(b)$$

$$y'(b) = \left(-\frac{1}{A}\int_a^b dy\, u(y)f(y)\right)v'(b).$$