A differential operator of the form

\[ L = \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \]

is said to be in **self-adjoint form**.

Why is that good? If \( L \) is self-adjoint, then

\[ L u = (p u')' + q u \]

and

\[
\int_a^b v^* L u \, dx = \int_a^b \left[ (p u')' + q u \right] v^* \, dx \\
= \left[ -pu' v^* + u q v^* \right]_a^b + \left[ v^* pu' \right]_a^b \\
= \int_a^b \left[ u p v^{\prime\prime} + u q v' \right] \, dx - \left[ p u' v^* \right]_a^b + \left[ v^* pu' \right]_a^b \\
= \int_a^b u L v^* \, dx + \left[ v^* pu' - v' pu \right]_a^b
\]

Suppose now that the boundary terms vanish. Then

\[
\int_a^b v^* L u \, dx = \int_a^b u L v^* \, dx = \int_a^b (L v^*) u \, dx.
\]
If in fact \( p \) and \( q \) are real so that \( L^* = L \), then

\[
\int_a^b v^* u \, dx = \int_a^b (x^*)^* u \, dx = \left( \int_a^b u^* x^r \, dx \right)^* \]

\[
\left\| u \right\| = \int_a^b v^* u \, dx \leq \int_a^b \left( \int_a^b v^* u \, dx \right)^* \]

\[
\langle v^* x^r u \rangle = \langle v^* x^r 1u \rangle = \langle u^* x^r v \rangle
\]

which we interpret as \( L = L^+ \) i.e., the operator \( L \) is hermitian. So a self-adjoint \( L \) with real \( p \) and \( q \) is hermitian.

The self-adjoint character of \( L \) is effective only if

\[
\left[ p(v^* u' - v^* u) \right]_a^b = 0, \quad \text{so one must choose}
\]

\( a \) and \( b \) carefully to fit each problem.

Usually, people require that all the functions \( u \) and \( v \) to which \( L \) is applied satisfy the boundary conditions

\[
\left. p v^* u' \right|_a^b = 0 = \left. p v^* v' \right|_b^b.
\]

Sometimes, one even \( v^* = 0 \) or \( v^r = 0 \),

\[
\left. p v^* u' \right|_a^b = 0 = \left. p v^* v' \right|_b^b, \quad \text{but these}
\]

extra conditions are not necessary.
In some cases $a$ and $b$ are taken to be $a = -\infty$ and $b = +\infty$, and it is assumed that

$$0 = P \nu(x) \text{ etc., at } x = \pm \infty.$$ 

One might imagine that self-adjoint operators are rare. But in fact one can multiply the generic 2nd-order differential operator

$$L u = P_0 u'' + p_1 u' + p_2 u$$

by

$$\frac{1}{P_0} e^{\int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt}$$

and get a self-adjoint $L$:

$$\frac{1}{P_0} e^{\int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt} L u = e^{\int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt} u'' + \frac{p_1}{P_0} e^{\int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt} u'$$

$$+ \frac{P_2}{P_0} e^{\int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt} u$$

$$= \left\{ \exp \left[ \int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt \right] u' \right\}' + \frac{P_2}{P_0} \exp \left[ \int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt \right] u$$

$$= (p u')' + g u \quad \text{where}$$

$$p = \exp \left[ \int_{p_0}^{x} \frac{P_1(t)}{P_0(t)} dt \right] \quad \text{and} \quad g = \frac{P_2}{P_0}.$$
So we always may cast a 2nd order differential operator into self-adjoint form, i.e., and if \( p, q \) are real, then it is Hermitean on functions \( u \) and \( v \) that satisfy the boundary conditions

\[
[p \left( v^* u' - u^* v' \right)]_a = 0.
\]

Eigen This and That

Suppose

\[
\lambda w^* u + L u = (p u')' + q u + \lambda w^* u = 0 \quad \text{i.e.}
\]

\[
(p(x) u'(x))' + q(x) u(x) + \lambda w(x) u(x) = 0,
\]

then \( \lambda \) is said to be the eigenvalue and \( w(x) \) is a known weight or density function. Here \( w(x) > 0 \) and \( w(x) = 0 \) only at isolated points. 'Eigen' means 'special' in Deutsch.

Legendre's eq. is a good example. It is

\[
0 = L y = (1-x^2)y'' - 2xy' + e(x) y
\]

\[
0 = (c_1 x^2 y)' + e(x) y
\]

As you will show when you do HW problem 8.5.5, series solutions exist for each
of the two solutions $k < 0$ and $k < 1$ of the indicial equation

$$k \ (k-1) = 0,$$

but the resulting series diverge for $x = \pm 1$ (i.e., $\theta = 0$ or $\pi$) unless $k$ is an integer. This is why orbital angular momentum is quantized. So the eigenvalue is $l (l+1)$ for integral $l$.

The deuteron is a spin-1 bound state of a neutron and a proton with binding energy about 2.2 MeV. It is mostly an $s$-state with some $d$-state mixed in. So the spins of the $n$ and $p$ are aligned. If $r$ is the relative distance in the reduced-mass formalism, then

$$-\frac{k^2}{2m} \Delta u + V(r) = E u, \quad V(r) = \begin{cases} V_0 & 0 < r < a \\ 0 & r \geq a \end{cases}$$

For an $s$-state boils down to this equation

$$u'' + k_1^2 u = 0 \quad u'' - k_2^2 u = 0$$

for $u(r) = r \psi(r)$ where $k_1 = \frac{2m}{\hbar^2} (E - V_0) > 0$ for $0 \leq r \leq a$ and

$$k_2^2 = \frac{2mE}{\hbar^2} \times 0$$

for $a < r$. 


So \( y(r) = \alpha \sin k_1 r + \beta \cos k_1 r \) for \( r < a \), but \( \beta = 0 \) to avoid a singularity at \( r = 0 \), in
\[
y(r) = \frac{u(r)}{r},
\]
outside the square well,
\[
u = A e^{k_2 r} + B e^{-k_2 r},
\]
but \( A = 0 \) so that \( y \) can be normalized.

When one matches the two solutions at the boundary, \( r = a \), by requiring that
\[
u_1(a) = \alpha \sin k_1 a = B e^{-k_2 a} = u_1(a) \quad \text{and} \quad u_1'(a) = \alpha k_1 \cos k_1 a = -k_2 B e^{-k_2 a} = u_1'(a)
\]
one finds that
\[
\tan k_1 a = -\frac{k_1}{k_2} = -\sqrt{\frac{E-V_0}{-E}}, \quad E < 0
\]
\( V_0 < 0, \left| V_0 \right| > |E| \),
\[
\tan \left( \sqrt{\frac{2m^2(E-V_0)}{k_2^2}} \right) = -\sqrt{\frac{E-V_0}{-E}}
\]
which has only discrete, quantized solutions for \( E < 0 \) and \( V_0 < 0, \left| V_0 \right| > |E| \).
L is 2nd order. What about first-order Hermitean operators? Some examples are

\[ \hat{p}^2 = \frac{\hbar}{i} \nabla \hat{p} \] in one dimension, \( p = \frac{\hbar}{i} \frac{d}{dx} \),

In this case,

\[ \int_a^b u^* \hat{p} u = \int_a^b \frac{\hbar}{i} u \frac{d}{dx} u^* + \frac{\hbar}{i} \int_a^b [u^* u] \]

\[ = \int_a^b u (\frac{\hbar}{i} \frac{d}{dx}) u^* + \frac{\hbar}{i} \int_a^b [u^* u] \]

So \( 0 = [u^* u] \)

\[ v^*(b) u(b) - v^*(a) u(a) \]

then

\[ \int_a^b u^* \hat{p} u = \left( \int_a^b u \frac{d}{dx} u^* \right) = \int_a^b (p^+ u^* u) \]

\[ = \int_a^b (p^- u^* u) \text{ so } p = p^+ \]

The trick is in the \( i \), and in the boundary conditions

\[ 0 = [v^* u] \]

which often are satisfied when \( a \to -\infty \),

\( b \to +\infty \), and both \( u \) and \( v \) are normalized.
Because of operators like $\mathbf{p}^2$, we generalize the notion of self-adjoint operators to hermitian operators, those for which
\[
\int_a^b v^* L u = \int_a^b (L^* v)^* u
\]
as long as $u$ and $v$ satisfy suitable boundary conditions at $x = a \& b$. We say $L = L^+$.

Suppose $L = L^+$, i.e., that $L$ is hermitian, and that
\[
L u_i + \lambda_i w u_i = 0 \quad i = 1, 2, \ldots.
\]
Then also
\[
L u_j + \lambda_j w u_j = 0
\]
so that
\[
(L u_j)^* + \lambda_j^* w u_j^* = 0.
\]
Note we take $w(x) = w^*(x)$ to be real. Then both
\[
u_j^* L u_i + \lambda_i^* w u_j^* u_i = 0
\]
and
\[
u_i (L u_j)^* + \lambda_j^* w u_j^* u_i = 0
\]
so that
\[
\int_a^b [u_j^* L u_i - u_i (L u_j)^*] = (\delta_j^i - \lambda_i^*) \int_a^b w(x) u_j^*(x) u_i(x) dx.
\]
Hence, since $L = L^+$, as long as the $u_i$'s satisfy the appropriate boundary conditions.
So,
\[ (\lambda_j - \lambda_i) \int_a^b dx \, w(x) \, u_j^*(x) u_i(x) = 0. \]

Set \( i = j \). Then
\[ (\lambda_j - \lambda_j) \int_a^b dx \, w(x) |u_i(x)|^2 = 0. \]

Since by assumption \( w(x) > 0 \) except at isolated points, it follows that
\[ \lambda_j = \lambda_i. \]

The eigenvalues of a Hamiltonian operator are real.

3. The top equation reads
\[ (\lambda_j - \lambda_i) \int_a^b dx \, w(x) \, u_j^*(x) u_i(x) = 0. \]

Thus the eigenfunctions \( u_i(x) \) and \( u_j(x) \) of different, unequal eigenvalues \( \lambda_j \neq \lambda_i \) must be orthogonal.

When two or more eigenfunctions do have the same eigenvalue, they are called degenerate. In the \( H \) atom, for instance, states with the same principal quantum number, \( n \), are degenerate to lowest order in the non-relativistic theory.
Suppose several \( u_i(x) \) all have the

same \( \lambda \)

\[
L u_i + \lambda w u_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, N.
\]

Then any linear combination of the \( u_i \)'s also
will satisfy

\[
L \left( \sum_{i=1}^{N} c_i u_i \right) + \lambda w \left( \sum_{i=1}^{N} c_i u_i \right) = 0
\]

because \( L \) is a linear differential operator.

So one may choose the \( c_i \)'s so as not
make mutually orthogonal linear combinations

\[
\psi_i = \sum_{j=1}^{N} c_{ij} u_j.
\]

We may even make them orthonormal

\[
d_i = \frac{\psi_i}{\left[ \int_{a}^{b} w(x) \right]^{1/2}}
\]

The Gram - Schmidt way:

Set \( \psi_i(x) = u_i(x) \)

\[
\phi_i(x) = \frac{\psi_i(x)}{\left[ \int_{a}^{b} \psi_i^2(x) w(x) \, dx \right]^{1/2}}
\]
For \( n = 2 \), we set

\[ \Psi_2(x) = u_2(x) + a_{21} \phi_1(x). \]

We want

\[ 0 = \int_a^b \psi_2(x) \phi_1^*(x) w(x) \]

\[ = \int_a^b u_2(x) \phi_1^*(x) w(x) + a_{21} \int_a^b |\phi_1(x)|^2 w(x). \]

So

\[ a_{21} = -\int_a^b u_2(x) \phi_1^*(x) w(x). \]

Thus

\[ \phi_2(x) = \frac{\Psi_2(x)}{\left[ \int_a^b |\Psi_2(x)|^2 w(x) \right]^{1/2}}. \]

Suppose now that \( \phi_1, \phi_2, \ldots, \phi_i \) are all orthonormal. We set

\[ \Psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^i a_{i+1,j} \phi_j(x), \quad \text{We set} \]

\[ 0 = \int_a^b \psi_{i+1}(x) \phi_j^*(x) w(x) \]

\[ 0 = \int_a^b u_{i+1}(x) \phi_j^*(x) w(x) + a_{i+1,j} \int_a^b |\phi_j(x)|^2 w(x). \]

So

\[ a_{i+1,j} = -\int_a^b u_{i+1}(x) \phi_j^*(x) w(x). \]
Finally

$$\phi_{i+1}(x) = \frac{\phi_{i+1}(x)}{\sqrt{\int_0^b |\phi_{i+1}(x)|^2 w(x) dx}}.$$

So to find the lilly, we write

$$\psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^i a_{i+1,j} \phi_j(x)$$

$$= u_{i+1}(x) - \sum_{j=1}^i \int_a^b u_{i+1}(x) \phi_j^*(x) w(x) \phi_j(x) dx$$

Now we set

$$P_j \psi_{i+1}(x) = \left[ \int_a^b u_{i+1}(x) \phi_j^*(x) w(x) dx \right] \phi_j(x), \quad \forall m$$

$$(1 - \sum_{j=1}^i P_j) \psi_{i+1}(x).$$

Example 9.2.1

Say $u_m(x) = x^m \quad m = 0, 1, 2, \ldots$

and the interval is $-1 \leq x \leq 1$ and $w(x) = 1$.

Now $u_0 = 1$, so $\psi_0 = 1$, so $\phi_0 = \frac{1}{\sqrt{2}}$.

$\psi_1 = x + a_{10} \frac{1}{\sqrt{2}}$ and so $a_{10} = -\int_{-1}^{1} x dx = 0$

and $\phi_1(x) = \frac{1}{\sqrt{2}} x$. 
\[ \psi_{2} = x^{2} + a_{20} \frac{1}{\sqrt{2}} + a_{21} \sqrt{\frac{3}{2}} \ x \quad \text{and} \]

\[ a_{20} = -\int_{-1}^{1} \frac{x^{2} dx}{\sqrt{2}} = -\frac{\sqrt{2}}{3} \]

\[ a_{21} = -\int_{-1}^{1} \sqrt{\frac{3}{2}} x^{3} dx = 0 \quad \text{and so} \]

\[ \psi_{2} = x^{2} - \frac{1}{3} \quad \text{and} \quad \phi_{-}^{2}(x) = \frac{5}{2} \frac{1}{2} (3x^{2} - 1). \]

Eventually

\[ \phi_{3}^{2}(x) = \sqrt{\frac{7}{2}} \cdot \frac{1}{2} (5x^{3} - 3x). \]

It turns out that these are the Legendre polynomials

\[ \phi_{n}(x) = \sqrt{\frac{2n+1}{2}} P_{n}(x), \]

apart from factors that reflect different normalization conditions.

Completeness

If any function \( f(x) \) in a certain space of functions \( S \) can be represented as

\[ f(x) = \sum_{n=0}^{\infty} a_{n} \phi_{n}(x) \]
in the sense that

\[ 0 = \lim_{N \to \infty} \int_a^b \left| f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \right|^2 w(x) dx, \]

then the set of functions \( \phi_n(x) \) is said to span that space \( S \) or to be complete in \( S \).

The coefficients \( a_n \) are

\[ \int_a^b f(x) \phi_n(x) w(x) dx = \sum_{j=0}^{\infty} a_j \phi_j(x) \phi_n^*(x) w(x) dx \]

\[ = \sum_{j=0}^{\infty} a_j \delta_{jn} = a_n \quad \text{so} \]

\[ a_n = \int_a^b f(x) \phi_n^*(x) w(x) dx. \]

But when \( Lu = (pu')' + qu \) with \( p \) and \( q \) real

\[ 0 = Lu + \lambda w u \quad \text{and} \quad w \geq 0, \]

then

\[ 0 = (Lu)^* + \lambda w u^* = Lu^* + \lambda w u^*. \]

So \( u^* \) is also an eigenfunction and so we may replace \( u \) by the real function \( \frac{u + u^*}{2} \) suitably normalized.
Typical spaces $S$ are the space $L_2$ of all square-integrable functions and the space $C$ of all piece-wise continuous functions. The proof that the eigenfunctions of any class of hermitian operators are complete in $L_2$ or $C$ is beyond the scope of this course.

But in the case of the sets of orthogonal polynomials — the Legendre polynomials and others listed in Table 9.3 — we can say more. These polynomials are equivalent to the powers of $x$, $x^n$ for $n \geq 0$. So we have half of a Laurent series or a whole power series.

If the $\phi_n$'s are complete for a space $S$ that includes the function $f$, then

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{where} \quad a_n = \int_a^b f(x)w(x)\phi_n^*(x)dx$$

$$f(x) = \sum_{n=0}^{\infty} \int_a^b f(y)w(y)\phi_n^*(y)dy \phi_n(x)$$

$$= \int_a^b \left[ \int_a^b f(y)w(y) \sum_{n=0}^{\infty} \phi_n^*(x)\phi_n^*(y) \right] dy \phi_n(x) S(x-y)$$

So

$$S(x-y) = (w(x)w(y))^{1/2} \sum_{n=0}^{\infty} \phi_n(x)\phi_n^*(y)$$

Completeness leads to a formula for $S(x-y)$. 
Bessel's inequality is

\[ 0 \leq \int_a^b w(x) f(x)^2 \, dx - \frac{\infty}{i=0} a_i \Phi_i^2 \]

\[ 0 \leq \int_a^b w(x) f(x)^2 \, dx - \int_a^b w(x) f(x) \sum q_i \Phi_i^2 \]

\[ - \int_a^b w(x) \Phi_i^2 \, dx \sum q_i \Phi_i^2 + \int_a^b w(x) \sum q_i \Phi_i^2 \Phi_j^2 \]

or

\[ 0 \leq \int_a^b w(x) f(x)^2 \, dx - \sum q_i a_i^2 - \sum q_i a_i^2 + \sum q_i a_i^2 \]

or

\[ \int_a^b w(x) f(x)^2 \, dx \geq \sum_{i=0}^{\infty} a_i \Phi_i^2 \]

In many cases, the absolute-value signs are superfluous.

\[ \text{Sohmanz's inequality} \]

Let \( \gamma = f + \lambda g \) so that

\[ 0 \leq \int_a^b (f^2 + \lambda g^2) \, dx = \int_a^b f^2 \, dx + \lambda \int_a^b g^2 \, dx + \lambda \int_a^b f g \, dx \]

Treating \( a \) and \( \lambda \) as independent variables, we get

\[ 0 \leq \int_a^b f^2 \, dx + \lambda \int_a^b g^2 \, dx \quad \text{and} \]

\[ 0 \leq \int_a^b f g \, dx + \lambda \int_a^b g f \, dx \]

\[ \int_a^b (f^2 + \lambda g^2) \, dx \geq \lambda \int_a^b g^2 \, dx \]

\[ \int_a^b (f^2 + \lambda g^2) \, dx \geq \lambda \int_a^b f^2 \, dx \]

\[ \int_a^b f g \, dx + \lambda \int_a^b g f \, dx \geq 0 \]
\[ 0 = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \]

With these values of \( a \) and \( b \), we get

\[ 0 \leq \int_a^b f(x)^2 \, dx = \int_a^b f(x) \, dx - \left( \int_a^b f(x) \, dx \right)^2 \left( \int_a^b g(x) \, dx \right)^2 \left( \int_a^b g(x) \, dx \right) \left( \int_a^b g(x) \, dx \right) \]

\[ \int_a^b f(x) \, dx \left( \int_a^b f(x) \, dx \right) + \int_a^b f(x) \, dx \int_a^b g(x) \, dx \]

so that

\[ \int_a^b f(x)^2 \, dx \int_a^b g(x)^2 \, dx \geq \left( \int_a^b f(x) + \int_a^b g(x) \, dx \right)^2 \]

The vector inequality is

\[ ||a \times b||^2 \leq a \cdot a \cdot b \cdot b\]

or in Dirac notation:

\[ (\Phi | 1) \leq (\Phi | \Phi) \cdot (1 | \Phi) \]

So when \( \Phi \) and \( \Psi \) are normalized, the probability of finding \( \Psi \) as \( \Phi \) is \( 1 \)

\[ P(\Phi, \Psi) \leq 1 \]
The review on pages 609-613 is worth reading.

Suppose $L = L^*$ is a hermitian operator with eigenfunctions $\phi_n$ and eigenvalues $\lambda_n$

$$L \phi_n + \lambda_n \phi_n = 0,$$

So $w(x) = 1$ here. Build the Green's function

$$G(x, y) = \sum_{n=0}^{\infty} \frac{\phi_n(x) \phi^*_n(y)}{\lambda_n - \lambda}.$$ 

See

$$(L + \lambda) G(x, y) = \sum_{n=0}^{\infty} \frac{(L + \lambda) \phi_n(x) \phi^*_n(y)}{\lambda_n - \lambda}$$ 

$$= \sum_{n=0}^{\infty} \frac{\lambda_n \phi_n(x) \phi^*_n(y)}{\lambda_n - \lambda}$$ 

$$= - \sum_{n=0}^{\infty} \phi_n(x) \phi^*_n(y) = - \delta(x-y).$$

So

$$(L + \lambda) G(x, y) = - \delta(x-y),$$

so if we have $L \psi + \lambda \psi = -f$, then we try

$$\psi(x) = \int d^3 y \ G(x, y) \rho(y)$$

so that

$$(L + \lambda) \psi(x) = \int d^3 x \ (L + \lambda) G(x, y) \rho(y) = -\int d^3 y \ \delta(x-y) \rho(y) = -f(x).$$
A more explicit treatment is available in one dimension. We will take $L$ to be well and self-adjoint

$$Lu = (pu')' + qu$$

and we want $y_0 \approx 0(x)$

$$Ly(y) + f(x) = 0,$$

Then

$$G(x, y) = -\frac{1}{A} \int_{x}^{y} u(y)v(x) \, dy,$$

here $A$ is a constant and

$$L u = L v = 0$$

and $u$ and $v$ respectively satisfy suitable boundary conditions at $x = a$ and $x = b$.

$$u(a)u'(a) = 0 \quad \text{or} \quad u(a) + \beta u'(a) = 0,$$

$$v(b)v'(b) = 0 \quad \text{or} \quad v(b) + \beta v'(b) = 0.$$

Set

$$y(x) = \int_{a}^{b} G(x, y) f(y) \, dy$$

$$= -\frac{1}{A} \int_{a}^{x} u(y)v(x) f(y) + \int_{x}^{b} u(x)v(y) f(y) \, dy - \frac{1}{A} \int_{a}^{b} u(x)v(y) f(y) \, dy$$

So

$$y'(x) = -\frac{u'(x)}{A} \int_{a}^{x} u(y) f(y) \, dy - \frac{u(x)}{A} \int_{x}^{b} v(y) f(y) \, dy$$

$$\left( -\frac{1}{A} u(x)v(x)f(x) + \frac{1}{A} u(x)v'(x)f(x) \right) = 0$$
\[ y''(x) = - \frac{v''(x)}{A} \int_x^A u'(x') f(x') dx' - \frac{u''(x)}{A} \int_x^A v(x') f(x') dx' \]

\[-\frac{1}{A} \left[ u(x) v'(x) - u'(x) v(x) \right] f(x). \]

Wronskian strikes again! Note that since \( u \) & \( v \) satisfy

\[ 0 = \Delta u = (pu')' + qu = \Delta v = (pv')' + qv = 0, \]

the Wronskian

\[ W = uv' - u'v \]

satisfies

\[ W' = uv'' - u''v = 0. \]

Now

\[ 0 = \Delta u \text{ implies } pu'' = -p'u - qu \text{ so } u'' = -\frac{p'u + qu}{p} \]

and \[ 0 = \Delta v \text{ implies } v'' = -\frac{p'v + qv}{p}. \] So

\[ W' = u \left( -\frac{p'u + qu}{p} \right) - \left( -\frac{p'v + qv}{p} \right) v = -\frac{q}{p} \left( uv' - v'u \right) = -\frac{q}{p} W \text{ so} \]

\[ \frac{W'}{W} = -\frac{q}{p} \left( \log W \right)' = -(\log p)' \]

So

\[ \log W = -\log p + c \]

\[ W = \frac{A}{p} \text{ so} \frac{W}{A} = \frac{1}{p} \]
So
\[ y''(x) = -\frac{v''(x)}{A} \int_a^x u(y) f(y)\,dy - \frac{u''(x)}{A} \int_a^x \nabla f(y)\,dy - \frac{f(x)}{\partial x} \]

So
\[ qy + py'' + p'y' = L y = -q\frac{v''}{A} - p\frac{v'}{A} \int_a^x u(y) f(y)\,dy - \frac{(qu' - pu'' - p' u)}{A} \int_a^x \nabla f(y)\,dy \]

\[ - f(x) \]

\[ Ly = -\frac{L v}{A} \int_a^x u(y) f(y)\,dy - \frac{L u}{A} \int_a^x \nabla f(y)\,dy - f(x) \]

But
\[ L v = L u = 0. \quad \text{So} \]

\[ Ly + f(x) = 0 \quad \text{or} \quad L y(x) + f(x) = 0. \]

Note that \( y(x) \) satisfies the same boundary conditions at \( x = a, b \) as \( u \) and \( v \):

\[ y(a) = -\frac{1}{A} \int_a^b dy u(y) f(y) = \left( -\frac{1}{A} \int_a^b dy v(y) f(y) \right) u(a) \]

\[ y'(a) = \left( -\frac{1}{A} \int_a^b dy v(y) f(y) \right) u'(a) \]

\[ y(b) = \left( -\frac{1}{A} \int_a^b dy u(y) f(y) \right) v(b) \]

\[ y'(b) = \left( -\frac{1}{A} \int_a^b dy u(y) f(y) \right) v'(b) \]