In the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$, the magnetostatic vector potential satisfies

$$\Delta\vec{A}(\vec{x}) = \nabla^2\vec{A}(\vec{x}) = -\frac{4\pi}{c}\vec{J}(\vec{x}).$$

Let

$$\vec{A}(\vec{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\tilde{A}(\vec{k})\,e^{-ik\cdot x} \quad \text{and}$$

$$\vec{J}(\vec{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\tilde{J}(\vec{k})\,e^{-ik\cdot x}.$$

Then

$$\nabla^2\vec{A}(x) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\tilde{A}(\vec{k})\,\nabla^2 e^{-ik\cdot x}$$

$$= \int\frac{d^3k}{(2\pi)^{3/2}}\,\tilde{A}(\vec{k})\,(-\vec{k}^2)\,e^{-ik\cdot x}$$

$$= -\frac{4\pi}{c}\vec{J}(\vec{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\,\tilde{J}(\vec{k})\left(-\frac{4\pi}{c}\right)e^{-ik\cdot x}.$$

So

$$-\vec{k}^2\,\hat{A}(\vec{k}) = -\frac{4\pi}{c}\,\tilde{J}(\vec{k}) \quad \text{or}$$

$$\hat{A}(k) = \frac{1}{c}\,\frac{4\pi}{\vec{k}^2}\,\tilde{J}(\vec{k}).$$

But if the Fourier transform $\hat{A}(\vec{k})$ is the product of $\hat{J}(\vec{k})$, the Fourier transform of the current density $\vec{J}'(\vec{x}')$, and $\frac{4\pi}{\vec{k}^2}$ is the Fourier transform of something else. What? Well, the something else, $G(\vec{x})$, must be

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{4\pi}{\vec{k}^2} e^{-i\vec{k}\cdot\vec{x}}$$

$$= \frac{8\pi^2}{(2\pi)^{3/2}} \int_{-1}^{1} d\mu \int_0^\infty dk \; e^{-ikr\mu} \qquad r = |\vec{x}|$$

$$= 2\sqrt{2\pi} \int_0^\infty dk \left(\frac{e^{-ikr} - e^{ikr}}{-ikr} \right)$$

$$= \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dk}{k} \frac{e^{ikr} - e^{-ikr}}{2i} = \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dk}{k} \sin kr$$

$$= \frac{4\sqrt{2\pi}}{r} \int_0^\infty \frac{dx}{x} \sin x = 4\sqrt{2\pi} \, \frac{\pi}{2} \, \frac{1}{r} = \frac{(2\pi)^{3/2}}{r}$$

by exercise 7.2.15.

So $G(\vec{x}) = \dfrac{(2\pi)^{3/2}}{|\vec{x}|}$ and

$$\tilde{\vec{A}}(k) = \frac{1}{c}\,\tilde{G}(\vec{k})\,\tilde{\vec{J}}(\vec{k})$$

whence

$$\vec{A}(x) = \frac{1}{c}\frac{1}{(2\pi)^{3/2}}\int d^3x'\; G(x-x')\,\vec{J}(x')$$

$$= \frac{1}{c}\frac{1}{(2\pi)^{3/2}}\int d^3x'\;\frac{(2\pi)^{3/2}}{|\vec{x}-\vec{x}'|}\,\vec{J}(\vec{x}')$$

$$= \frac{1}{c}\int\frac{d^3x'}{|\vec{x}-\vec{x}'|}\,\vec{J}(\vec{x}').$$

The 3-d convolution theorem is

$$f * g\,(x) = \frac{1}{(2\pi)^{3/2}}\int d^3y\; f(\vec{x}-\vec{y})\,g(\vec{y})$$

$$= \frac{1}{(2\pi)^{3/2}}\int d^3y\; g(\vec{x}-\vec{y})\,f(\vec{y}) \quad \text{and}$$

$$\widetilde{f*g}(\vec{k}) = \tilde{f}(\vec{k})\,\tilde{g}(\vec{k}).$$

Going back to the differential equation

$$\nabla^2 \vec{A}(\vec{x}) = -\frac{4\pi}{c} \vec{J}(\vec{x}),$$

we see that the relation

$$\nabla^2 \vec{A}(\vec{x}) = \frac{1}{c} \int d^3x' \; \nabla^2 \frac{1}{|x - x'|} \; \vec{J}(x')$$

$$= -\frac{4\pi}{c} \vec{J}(\vec{x})$$

implies that

$$-\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \; \delta^{(3)}(\vec{x} - \vec{x}').$$

So

$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi |\vec{x} - \vec{x}'|}$$

is the Green's function for $-\nabla^2$:

$$-\nabla^2 \frac{1}{4\pi |\vec{x} - x'|} = \delta^{(3)}(\vec{x} - \vec{x}').$$

Yet another representation of Dirac's delta function!

If $\quad -\nabla^2 G(\vec{x}, \vec{x}') = \delta^{(3)}(\vec{x} - \vec{x}')$, then

$$G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \frac{1}{\vec{k}^2}$$

for then

$$-\nabla^2 G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} = \delta^{(3)}(\vec{x} - \vec{x}').$$

And

$$G(\vec{x}, \vec{x}') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\vec{k}^2} = \frac{1}{4\pi|\vec{x} - \vec{x}'|}$$

according to the integration on p. 341.

So

$$-\nabla^2 \frac{1}{4\pi|\vec{x} - \vec{x}'|} = \delta^{(3)}(\vec{x} - \vec{x}').$$

Note
$$G(\vec{x}, \vec{x}') = \frac{1}{4\pi|\vec{x} - \vec{x}'|}$$
is the Green's function, while on p. 341,

$$G(\vec{x}) = \frac{(2\pi)^{3/2}}{r}.$$

This delta-function helps us solve

$$\nabla\cdot E = 4\pi\rho .$$

First, in the Coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0$, the divergence of

$$-\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \quad \text{is} \quad \nabla\cdot E = -\nabla^2\phi$$

and so

$$-\nabla^2\Phi = 4\pi\rho .$$

We now form the convolution

$$\Phi(\vec{x}) = \int \frac{\rho(\vec{x}\,')}{|\vec{x} - \vec{x}\,'|}\, d^3x'$$

and note that

$$-\nabla^2\Phi(x) = \int \rho(\vec{x}\,')\left(-\nabla^2\frac{1}{|\vec{x}-\vec{x}\,'|}\right) d^3x'$$

$$= 4\pi\int \rho(\vec{x}\,')\,\delta^{(3)}(\vec{x}-\vec{x}\,')\, d^3x'$$

$$= 4\pi\,\rho(\vec{x}).$$

So in the Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$,

$$\Phi(\vec{x},t) = \int \frac{\rho(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|}\, d^3x'$$

even in the time-dependent case,

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t},$$

and

$$\nabla \cdot E = -\nabla^2 \phi = 4\pi\rho .$$

In the static case, $\vec{A}(\vec{x})$ is given by

$$\vec{A}(\vec{x}) = \frac{1}{c}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' , \quad \text{but}$$

what about the time-dependent $\vec{A}(\vec{x},t)$?

We start with $B = \nabla \times A$ and

$$\overrightarrow{\nabla \times B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

$$\left[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})\right]_i = \epsilon_{ijk}\,\partial_{k\ell m}\,\partial_j \partial_\ell A_m = \frac{4\pi}{c} J_i + \frac{1}{c}\frac{\partial E_i}{\partial t}$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\,\partial_j\partial_\ell A_m = -\partial_j^2 A_i \quad \text{since } \nabla\cdot A = 0$$

so

$$-\partial_j^2 A_i = \frac{4\pi}{c} J_i + \frac{1}{c}\frac{\partial}{\partial t}\left(-\partial_i\phi - \frac{1}{c}\frac{\partial A_i}{\partial t}\right) \quad \text{or}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A_i = \frac{4\pi}{c}\left(J_i - \frac{1}{4\pi}\partial_i\dot{\phi}\right) \equiv \frac{4\pi}{c}\vec{J}_t$$

where $\nabla\cdot J_t = \nabla\cdot J - \frac{1}{4\pi}\nabla^2\dot{\phi} = \nabla\cdot J + \dot{\rho} = 0$.
$\vec{J}_t$ is the transverse current density.
The equation $0 = \vec{\nabla}\cdot\vec{J} + \dot{\rho}$ is called current conservation.

So

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) A_i = \frac{4\pi}{c} J_{t\,i} \quad \text{or}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) \vec{A} = \Box \vec{A} = \frac{4\pi}{c} \vec{J}_t \, .$$

We seek a Green's function for $\Box$:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) G(x,t;x',t') = 4\pi\, \delta^3(x-x')\, \delta(t-t')$$

So we write

$$G(\vec{x},t;\vec{x}',t') = \int d^3k \int_{-\infty}^{\infty} d\omega \, e^{ik(x-x') - i\omega(t-t')} g(\vec{k},\omega)$$

and get

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G = \int d^3k \int d\omega \left(\vec{k}^2 - \frac{\omega^2}{c^2}\right) e^{i\vec{k}\cdot(\vec{x}-\vec{x}') - i\omega(t-t')} g(\vec{k},\omega)$$

$$= 4\pi \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\vec{k}\cdot(\vec{x}-\vec{x}') - i\omega(t-t')}$$

So

$$g(\vec{k},\omega) = \frac{1}{4\pi^3} \frac{1}{\vec{k}^2 - \frac{\omega^2}{c^2}} ,$$

but we need to interpret the singularity at

$$k^2 = \frac{\omega^2}{c^2} .$$

Our formula for $A(\vec{x},t)$ in the Coulomb gauge is

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x'\,dt'\; G(\vec{x},t;\vec{x}',t')\,\vec{J}_t(x',t').$$

So we want G to vanish when $t' > t$. So we put the poles in the LHP:

$$G(\vec{x},t;\vec{x}',t') = \int d^3k \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\vec{k}\cdot\vec{r} - i\omega\tau}}{4\pi^3}\,\frac{1}{\left(k^2 - \frac{1}{c^2}(\omega+i\epsilon)\right)^2}$$

where $\vec{r} = \vec{x} - \vec{x}'$ and $\tau = t - t'$. For $\tau < 0$,

$$\int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega\tau}}{k^2 - \frac{1}{c^2}(\omega+i\epsilon)^2} = \int \frac{dz\, e^{+i|\tau|z}}{k^2 - \frac{1}{c^2}(z+i\epsilon)^2} = 0.$$

So $G(r,\tau) = 0$ for $\tau < 0$. For $\tau > 0$,

$$\int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega\tau}}{k^2 - \frac{1}{c^2}(\omega+i\epsilon)^2} = \int \frac{dz\, e^{-iz\tau}}{k^2 - \frac{1}{c^2}(z+i\epsilon)^2}$$

$$= -c^2 \oint \frac{dz\, e^{-iz\tau}}{(z+i\epsilon-kc)(z+i\epsilon+kc)}$$

$$= + c^2 2\pi i \left[\frac{e^{-i\tau(-kc)}}{-2kc} + \frac{e^{-i\tau kc}}{2kc} \right]$$

$$= \frac{2\pi}{c^3} \frac{1}{k} \left(\frac{e^{i\tau kc} - e^{-i\tau kc}}{2i} \right) .$$ So

$$G(\vec{r},\tau) = \frac{c}{2\pi^2} \int d^3k \, \frac{1}{k} \sin(kc\tau) \, e^{i\vec{k}\cdot\vec{r}}$$

$$= \frac{c}{\pi} \int_{-1}^{1} d\mu \int_0^\infty dk \; k \sin(kc\tau) \, e^{ikr\mu}$$

$$= \frac{c}{\pi} \int_0^\infty dk \; k \sin(kc\tau) \frac{e^{ikr} - e^{-ikr}}{ikr}$$

$$= \frac{2c}{\pi r} \int_0^\infty dk \; \sin(kr) \sin(ck\tau) .$$

Let $x = ck$. The integrand is even in x

$$G(\vec{r},\tau) = \frac{1}{\pi r} \int_{-\infty}^{\infty} dx \left(\frac{e^{i\frac{xr}{c}} - e^{-i\frac{xr}{c}}}{2i} \right) \left(\frac{e^{ix\tau} - e^{-ix\tau}}{2i} \right)$$

$$= \frac{1}{4\pi r} \int_{-\infty}^{\infty} dx \left[e^{i(\tau - \frac{r}{c})x} - e^{i(\tau + \frac{r}{c})x} + e^{i(\frac{x}{c} - \tau)x} - e^{-i(\tau + \frac{r}{c})x} \right]$$

which is just a sum of

delta functions:

$$G(\vec{r}, \tau) = \frac{1}{r}\left[\delta\left(\tau - \frac{r}{c}\right) - \delta\left(\tau + \frac{r}{c}\right)\right].$$

But here $\tau > 0$, so

$$G(\vec{r}, \tau) = \frac{\delta(\tau - r/c)}{r}$$

or

$$G(\vec{x}, t; \vec{x}', t') = \frac{\delta(t - t' - |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|}.$$

This retarded Green's function satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) G(\vec{x}, t; \vec{x}', t') = 4\pi \delta^3(\vec{x} - \vec{x}')\, \delta(t - t').$$

So in the Coulomb gauge, $\nabla \cdot A = 0$, the time-dependent vector potential $\vec{A}(\vec{x}, t)$ is

$$\vec{A}(\vec{x}, t) = \frac{1}{c}\int d^3x'\, dt' \, \frac{\delta(t - t' - |x - x'|/c)}{|\vec{x} - \vec{x}'|} \vec{J}_t(x', t')$$

and it satisfies

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A}(\vec{x}, t) = \frac{4\pi}{c}\vec{J}_t(x, t).$$

If we do the t' integral, then the vector potential

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d^3x' \, \frac{\vec{J}_t(x', t - |\vec{x}-\vec{x}'|/c)}{|\vec{x}-\vec{x}'|}$$

depends upon the current density J at the earlier time

$$t' = t - \frac{|\vec{x}-\vec{x}'|}{c}.$$

But, as we saw on page 345, the scalar potential in the Coulomb gauge, $\nabla \cdot A = 0$, is "instantaneous":

$$\Phi(\vec{x},t) = \int \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \, d^3x'.$$

The Coulomb-gauge condition

$$\vec{\nabla}\cdot\vec{A}(\vec{x},t) = 0$$

is not Lorentz invariant, so electrodynamics in the Coulomb gauge looks as though it violates relativity. But it doesn't. Electrodynamics in the Coulomb gauge does respect special relativity.

The Coulomb gauge is also called the radiation gauge.

The Lorentz gauge condition

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = \partial_\mu A^\mu = 0$$

is itself Lorentz invariant.

Why are convolutions ubiquitous? Because space-time is homogeneous. The convolution

$$A(x) = \int d^4x' \, J(x') \, f(x-x')$$

respects the homogeneity of space-time. Suppose

$$A_2(x) = \int d^4x' \, J_2(x') \, f(x-x')$$
$$A_1(x) = \int d^4x' \, J_1(x') \, f(x-x') \qquad \text{and}$$
$$J_2(x) = J_1(x+b)$$

ie., the two sources differ by a translation b.
Then

$$A_2(x) = \int d^4x' \, J_1(x'+b) \, f(x-x')$$
$$= \int d^4x' \, J_1(x') \, f(x-(x'-b))$$
$$= \int d^4x' \, J_1(x') \, f(x+b-x') = A_1(x+b)$$

ie., the fields differ only by the same translation b. So the symmetry that space time is homogeneous leads to convolutions. The dynamics might be

$$\mathcal{D}(-i\nabla)\, A(x) = J(x),$$

Let

$$f(x-x') = \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{ik\cdot(x-x')}}{\mathcal{D}(k)} \quad \text{so that}$$

$$\mathcal{D}(-i\nabla)\, f(x-x') = \delta^{(4)}(x-x').$$

The inner product of $|\vec{p}\,\rangle$ with $|\vec{x}\,\rangle$ is

$$\langle \vec{x}\,'|\vec{p}\,'\rangle = \frac{e^{i\vec{x}\,'\cdot\vec{p}\,'/\hbar}}{(2\pi\hbar)^{3/2}},$$

in which $\vec{x}\,'$ and $\vec{p}\,'$ are the eigenvalues (ev) of the operators $\vec{x}$ and $\vec{p}$

$$\vec{x}\,|\vec{x}\,'\rangle = \vec{x}\,'|x'\rangle$$

$$\vec{p}\,|\vec{p}\,'\rangle = \vec{p}\,'\,|p'\rangle.$$

These eigenvectors $|\vec{x}\,'\rangle$ and $|\vec{p}\,'\rangle$ are complete

$$1 = \int d^3x'\, |\vec{x}\,'\rangle\langle\vec{x}\,'| = \int d^3p'\, |\vec{p}\,'\rangle\langle\vec{p}\,'|$$

so that

$$1 = \int d^3x'\, |\vec{x}\,'\rangle\langle\vec{x}\,'| \int d^3p'\, |\vec{p}\,'\rangle\langle\vec{p}\,'| \int d^3x''\, |\vec{x}\,''\rangle\langle\vec{x}\,''|$$

$$= \int d^3x'\, d^3p'\, d^3x''\, |\vec{x}\,'\rangle\langle\vec{x}\,''|\, \frac{e^{i(\vec{x}\,'-\vec{x}\,'')\cdot\vec{p}\,'/\hbar}}{(2\pi\hbar)^3}$$

Let $\vec{k} = p'/\hbar$

$$1 = \int d^3x'\, d^3x''\, d^3k\, \frac{e^{i(\overrightarrow{x'-x''})\cdot\vec{k}}}{(2\pi)^3}\, |\vec{x}\,'\rangle\langle\vec{x}\,''| = \int d^3x'\, d^3x''\, |\vec{x}\,'\rangle\langle x''|\, \delta^{(3)}(\overrightarrow{x'-x''})$$

$$= \int d^3x'\, |\vec{x}\,'\rangle\langle\vec{x}\,'| = 1.$$

Consider the gaussian wave packet

$$\langle \vec{x}' | \psi \rangle = \psi(\vec{x}') = \left(\frac{1}{\pi^{1/4}\sqrt{d}} \right)^3 e^{i\vec{k}\cdot\vec{x}' - \frac{\vec{x}'^2}{2d^2}}$$

which is a plane wave of wave number $\vec{k} = \vec{p}/\hbar$ modulated by a gaussian profile. The probability density

$$P(\vec{x}') = |\langle \vec{x}' | \psi \rangle|^2$$

$$= \left(\frac{1}{d\sqrt{\pi}} \right)^3 e^{-\frac{\vec{x}'^2}{d^2}}$$

drops sharply away from the origin $\vec{x}' = \vec{0}$. So we have a particle of momentum nearly $\hbar\vec{k}'$ nearly at $\vec{x}' = \vec{0}$.

The mean value of $\vec{x}'$ vanishes

$$\langle \vec{x} \rangle = \int d^3x' \; P(\vec{x}')\vec{x}' = \int \frac{d^3x' \; \vec{x}'}{(d\sqrt{\pi})^3} e^{-\vec{x}'^2/d^2} = 0,$$

but

$$\langle \vec{x}^2 \rangle = \int d^3x' \; P(\vec{x}') \vec{x}'^2$$

$$= \frac{1}{(d\sqrt{\pi})^3} \int d^3x' \; \vec{x}'^2 e^{-\vec{x}'^2/d^2}$$

$$= \frac{1}{(d\sqrt{\pi})^3}\left(-\frac{d}{d\,d^{-2}}\right)\int d^3x'\, e^{-x'^2/d^2}$$

$$= \frac{1}{(d\sqrt{\pi})^3}\left(\frac{d}{d(-d^{-2})}\right) d^3 \int d^3y\, e^{-\vec{y}^{\,2}}$$

let $\alpha = \frac{1}{d^2}$

$$= \frac{\int d^3y\, e^{-\vec{y}^{\,2}}}{(d\sqrt{\pi})^3}\left(-\frac{d}{d\alpha}\alpha^{-3/2}\right)$$

$$= \frac{3}{2}\alpha^{-5/2}\int \frac{d^3y\, e^{-\vec{y}^{\,2}}}{(d\sqrt{\pi})^3}$$

$$= \frac{3}{2}\frac{d^5}{d^3}\int \frac{d^3y}{(\sqrt{\pi})^3}\, e^{-\vec{y}^{\,2}}$$

Recall

$$\left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^2 = \int dx\,dy\, e^{-x^2-y^2} = \int 2\pi r\,dr\, e^{-r^2} = \pi$$

So

$$\langle \vec{x}'^2 \rangle = \frac{3}{2} d^2 \quad \text{and also}$$

$$\int d^3x'\, P(x') = \int \frac{d^3x'}{(d\sqrt{\pi})^3}\, e^{-\frac{\vec{x}'^2}{d^2}} = \int \frac{d^3y}{\pi^{3/2}}\, e^{-y^2} = 1.$$

So $P(x')$ is normalized.

What is $\langle \vec{p} \rangle$? As HW 4, problem 1, show

$$\langle \vec{p} \rangle = \int d^3\vec{x}' \, \psi^*(\vec{x}') \, \frac{\hbar}{i} \nabla \psi(\vec{x}') = \hbar k = \vec{p}'$$

and

$$\langle \vec{p}^{\,2} \rangle = \int d^3x' \, \psi^*(\vec{x}') \left[-\hbar^2 \nabla^2 \psi(\vec{x}') \right]$$

$$= \frac{3\hbar^2}{2d^2} + \hbar^2 \vec{k}^2 = \frac{3\hbar^2}{2d^2} + \vec{p}'^{\,2}.$$

It follows then that

$$\langle (\Delta x)^2 \rangle = \langle (\vec{x} - \langle \vec{x}' \rangle)^2 \rangle$$

$$= \langle \vec{x}^{\,2} - 2\langle \vec{x} \rangle^2 + \langle \vec{x} \rangle^2 \rangle$$

$$= \langle \vec{x}^{\,2} \rangle - \langle \vec{x} \rangle^2 = \langle \vec{x}^{\,2} \rangle = \frac{3}{2} d^2$$

and that

$$\langle (\Delta \vec{p})^2 \rangle = \langle (\vec{p}' - \langle \vec{p} \rangle)^2 \rangle$$

$$= \langle \vec{p}^{\,2} \rangle - \langle \vec{p} \rangle^2$$

$$= \frac{3}{2} \frac{\hbar^2}{d^2} + \hbar^2 \vec{k}'^2 - \hbar^2 \vec{k}^2$$

$$= \frac{3}{2} \frac{\hbar^2}{d^2}.$$

Let's look at a single degree of freedom, say x_1 and p_1. Then

$$\langle(\Delta x_1)^2\rangle = \frac{1}{2} d^2$$

and

$$\langle(\Delta p_1)^2\rangle = \frac{1}{2}\frac{\hbar^2}{d^2},$$

so the product is

$$\langle(\Delta x_1)^2\rangle\langle(\Delta p_1)^2\rangle = \frac{1}{4}\hbar^2 \quad \text{or}$$

$$\Delta x_1 \, \Delta p_1 = \frac{\hbar}{2}.$$

The momentum-space amplitude is

$$\langle \vec{p}' | \psi \rangle = \int d^3x' \, \langle \vec{p}' | \vec{x}' \rangle \langle x' | \psi \rangle$$

$$= \int \frac{d^3x' \, e^{-i p'\cdot x'/\hbar}}{(2\pi\hbar)^{3/2}} \frac{1}{(\pi^{1/4}\sqrt{d})^3} e^{i k\cdot x' - x'^2/(2d^2)}$$

$$= \frac{e^{-\frac{d^2}{2\hbar^2}(p'-\hbar k)^2}}{(2\pi\hbar)^{3/2}(\pi^{1/4}\sqrt{d})^3} \int d^3x' \, e^{-\frac{(\vec{x}' + i d^2(\vec{p}'/\hbar - \vec{k}))^2}{2d^2}}$$

Now the single integral

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{(x+i\alpha)^2}{2d^2}} = \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2d^2}} = d\sqrt{2}\int_{-\infty}^{\infty} dy\, e^{-y^2} = d\sqrt{2\pi}$$

So

$$\langle \vec{p}' | \psi \rangle = \frac{d^{3/2}}{\hbar^{3/2}} \frac{1}{\pi^{3/4}} e^{-\frac{d^2}{2\hbar^2}(\vec{p}' - \hbar \vec{k})^2}$$

What is $\psi(\vec{x}', t)$? We take

$$H = \frac{\vec{p}^2}{2m} \quad \text{where} \quad \vec{p}\,|\vec{p}'\rangle = \vec{p}'\,|\vec{p}'\rangle.$$

$$\langle \vec{x}' | \psi, t \rangle = \langle \vec{x}' | e^{-iHt/\hbar} | \psi \rangle$$

$$= \langle \vec{x}' | e^{-i\frac{Ht}{\hbar}} \int d^3p'\, |\vec{p}'\rangle\langle \vec{p}' | \psi \rangle$$

$$= \int d^3p'\, \langle \vec{x}' | p' \rangle\, e^{-i\frac{p'^2 t}{2m\hbar}} \langle \vec{p}' | \psi \rangle$$

$$= \int d^3p'\, \frac{e^{i\vec{x}\cdot\vec{p}'/\hbar}}{(2\pi\hbar)^{3/2}}\, e^{-i\frac{p'^2 t}{2m\hbar}} \frac{d^{3/2}}{\hbar^{3/2}} \frac{1}{\pi^{3/4}} e^{-\frac{d^2}{2\hbar^2}(\vec{p}' - \hbar\vec{k})^2}$$

HW 4, Problem 2, Find $\langle \vec{x}' | \psi, t \rangle$.

The Laplace transform $f(s)$ of a function $F(t)$ is

$$f(s) = \int_0^\infty dt \; e^{-st} F(t) = \mathcal{L}\{F(t)\}.$$

So if $F(t) = 1$, then

$$f(s) = \int_0^\infty dt \; e^{-st} = \frac{e^{-st}}{-s}\Big]_0^\infty = \frac{1}{s} .$$

If $F(t) = e^{kt}$, then

$$f(s) = \int_0^\infty dt \; e^{-st} e^{kt} = \frac{1}{s-k} \quad \text{for } s > k.$$

So if $F(t) = \operatorname{ch} kt$, then

$$f(s) = \int_0^\infty dt \; e^{-st} \left(\frac{e^{kt} + e^{-kt}}{2} \right)$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k} \right) = \frac{s}{s^2 - k^2} .$$

And if $F(t) = \operatorname{sh} kt$, then

$$f(s) = \int_0^\infty dt \; e^{-st} \left(\frac{e^{kt} - e^{-kt}}{2} \right) = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k} \right) = \frac{k}{s^2 - k^2} .$$

Similarly if $F(t) = \cos kt$

$$f(s) = \int_0^\infty dt\, e^{-st} \frac{e^{ikt} + e^{-ikt}}{2} = \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$

$$= \frac{s}{s^2+k^2} \qquad \text{and}$$

if $F(t) = \sin kt$, then

$$f(s) = \int_0^\infty dt\, e^{-st} \left(\frac{e^{ikt} - e^{-ikt}}{2i}\right) = \frac{1}{2i}\left(\frac{1}{s-ik} - \frac{1}{s+ik}\right)$$

$$= \frac{k}{s^2+k^2}.$$

Since

$$\frac{1}{s} = \int_0^\infty dt\, e^{-st}$$

$$-\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} = \int_0^\infty dt\, t\, e^{-st}$$

$$\left(-\frac{d}{ds}\right)^2 \frac{1}{s} = \frac{2}{s^3} = \int_0^\infty dt\, t^2 e^{-st}$$

$$\left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}} = \int_0^\infty dt\, t^n e^{-st}, \quad \text{for } s>0.$$

If $\quad f(s) = \int_0^\infty dt\, e^{-st} F(t) = \mathcal{L}\{F(t)\}$, then

$$\mathcal{L}\{F'(t)\} = \int_0^\infty dt\, e^{-st} \frac{d}{dt} F(t)$$

$$= \int_0^\infty dt \frac{d}{dt}\left[e^{-st} F(t)\right] - F(t)\frac{d}{dt} e^{-st}$$

$$= \int_0^\infty dt\; s\, e^{-st} F(t) \;-\; F(0)$$

$$= s\,\mathcal{L}\{F(t)\} - F(0),$$

So

$$\mathcal{L}\{F''(t)\} = s\,\mathcal{L}\{F'(t)\} - F'(0)$$

$$= s\left[s\,\mathcal{L}\{F(t)\} - F(0)\right] - F'(0)$$

$$= s^2\mathcal{L}\{F(t)\} - sF(0) - F'(0).$$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty dt\, e^{-st}\,\delta(t-t_0) = e^{-st_0} \qquad t_0 \geq 0$$

$$\mathcal{L}\{\delta(t)\} = 1.$$

How does one invert

$$f(s) = \int_0^\infty dt\, e^{-st} F(t) \quad ?$$

Well, consider the integral

$$\int_{-\infty}^{\infty} f(is) \frac{ds}{2\pi} e^{ist} = \int_{-\infty}^{\infty} ds \int_0^\infty \frac{dt'}{2\pi} e^{is(t-t')} F(t') = F(t)$$

So the inverse is

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{ist} f(is) \, .$$

But a Laplace transform $f(s)$ tends to be smoother as Re s increases. So one may need to use

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{(x+is)t} f(x+is) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \int_0^\infty dt'\, e^{(x+is)t} e^{-(x+is)t'} F(t')$$

$$= \int_0^\infty dt'\, e^{x(t-t')} \delta(t-t') F(t') = F(t).$$

The inversion formula

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{(x+is)t} f(x+is)$$

is called a Bromwich integral or a Fourier-Mellin integral.

Homogeneous partial differential equations often possess very simple solutions. For example, the wave equation

$$(\Box - m^2c^2)\,\phi(x,t) = 0 = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - m^2c^2\right)\phi(x,t) = 0$$

may be solved by writing $\phi(x,t) = F(\vec{p}\cdot\vec{x} - Et)$ where F is any harmonic function $F'' = -F$ and $E^2 = \vec{p}^{\,2}c^2 + m^2c^4$.

$$\vec{\nabla}\phi(x,t) = F'\,\vec{\nabla}(p\cdot x - Et) = \vec{p}\,F'$$

and so

$$\Delta\phi = \vec{p}^{\,2}F'' = -\vec{p}^{\,2}F$$

Also

$$\frac{1}{c}\frac{\partial}{\partial t}\phi = -\frac{E}{c}F'$$

and so

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = +\frac{E^2}{c^2}F'' = -\frac{E^2}{c^2}F$$

So

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - m^2\right)\phi(x,t) = -\vec{p}^{\,2}F + \frac{E^2}{c^2}F - m^2c^2F$$

$$= \left(\frac{E^2}{c^2} - p^2 - m^2c^2\right)F = 0$$

as long as

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4.$$

Note that this trick works for every $(\vec{p}, E)$ that is "on the mass shell"

$$E^2 = c^2 \vec{p}^2 + m^2 c^4 .$$

One needs to use all these $(\vec{p}, E)$'s to get a general solution. Thus the vector potential $\vec{A}(x,t)$ is (here $m=0$)

$$\vec{A}(\vec{x},t) = \sum_{\substack{c^2p^2=E^2 \\ s=1}}^{2} \left(\frac{\hbar^2 c^2}{2VE}\right)^{\frac{1}{2}} \Big[\vec{\epsilon}(p,s)\, e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar}\, a(p,s) + \vec{\epsilon}^{\,*}(p,s)\, e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar}\, a^{+}(p,s) \Big]$$

in quantum electrodynamics. Take the mean value of $\vec{A}(\vec{x},t)$ in a coherent state $|\alpha\rangle$

$$a(p,s)\,|\alpha\rangle = \alpha(p,s)\,|\alpha\rangle \qquad (\langle\alpha|\alpha\rangle = 1)$$

and get the classical field:

$$\langle\alpha|\,\vec{A}(\vec{x},t)\,|\alpha\rangle = \sum_{\substack{c^2p^2=E^2 \\ s=1}}^{2} \left(\frac{\hbar^2 c^2}{2VE}\right)^{\frac{1}{2}} \Big[\vec{\epsilon}(p,s)\, \alpha(p,s)\, e^{i(\vec{p}\cdot\vec{x}-Et)/\hbar} + \vec{\epsilon}^{\,*}(p,s)\, \alpha^{*}(p,s)\, e^{-i(\vec{p}\cdot\vec{x}-Et)/\hbar} \Big].$$

Note that $a(p,s)$ destroys a photon of $\vec{p}, s$, while $a^{+}(p,s)$ makes one. And $\exp(\,i(p\cdot x - Et)/\hbar\,)$ is a plane wave. Their product $a(p,s)\exp[i(p\cdot x - Et)/\hbar]$ encodes the "wave-particle duality" of quantum mechanics.

Many kinds of homogeneous PDEQ's can be solved by such tricks.

A related trick is the JWKB approximation

$$\psi(x,t) = \sqrt{\rho(x,t)}\; e^{i\frac{S(\vec{x},t)}{\hbar}} .$$

The fast behavior here is $S/\hbar$, not $\rho(\vec{x},t)$. So

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{gives}$$

$$\frac{(\vec{\nabla}S)^2}{2m} + V + \frac{\partial S}{\partial t} = 0$$

which is the Hamilton-Jacobi equation. For a stationary state, one expects

$$S(\vec{x},t) = W(\vec{x}) - Et \quad \text{where}$$

$W(\vec{x})$ is Hamilton's characteristic function for which

$$\frac{(\nabla W)^2}{2m} + V = E .$$

Let us switch to natural units $\hbar = c = 1$ to discuss non-linear PDEQ's.

$$\Box \Phi = V'(\Phi)$$

Suppose $\Phi(x,t) = \phi(p \cdot x - Et)$, then

$$(p^2 - E^2)\,\phi'' = V'(\phi)$$ and so

$$(p^2 - E^2)\,\phi''\phi' = V'(\phi)\,\phi'$$

$$(p^2 - E^2)\,\frac{\phi'^2}{2} = V(\phi) - \mathcal{E}$$ where $\mathcal{E}$ is a constant of integration

$$\mathcal{E} = m^2 \frac{\phi'^2}{2} + V(\phi)$$

This is like ordinary classical mechanics.

$$\phi'^2 = \frac{2}{m^2}\left(\mathcal{E} - V(\phi)\right)$$

Let $z = p \cdot x - Et$, then

$$\frac{d\phi}{dz} = \frac{\sqrt{2}}{m}\sqrt{\mathcal{E} - V(\phi)}$$ so

$$\int \frac{d\phi}{\sqrt{\mathcal{E} - V(\phi)}} = \int \frac{\sqrt{2}}{m}\,dz = \frac{\sqrt{2}}{m}(z - z_0)$$

where z_0 is another integration constant.

The simplest case is the time-independent case for which $\epsilon = 0$. Then if $p = (1,0,0)$ & $\mathcal{E} = 0$,

$$\frac{\phi'^2}{2} = V(\phi).$$

As an example, let's take

$$V(\phi) = \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2.$$

Then

$$\frac{\phi'^2}{2} = \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2 \quad \text{or}$$

$$\phi' = \pm\lambda\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right) \quad \text{whence}$$

$$\int \frac{d\phi}{\phi^2 - \frac{\mu^2}{\lambda^2}} = -\frac{\lambda}{\mu}\tanh^{-1}\left(\frac{\phi\lambda}{\mu}\right) = \pm\lambda(x - x_0)$$

or

$$\tanh^{-1}\left(\phi\frac{\lambda}{\mu}\right) = \mp\mu(x_0 - x) \quad \text{or}$$

$$\phi\frac{\lambda}{\mu} = \mp\tanh\mu(x_0 - x) \quad \text{or}$$

$$\phi(x) = \pm\frac{\mu}{\lambda}\tanh\mu(x - x_0)$$

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This is a soliton in the loose sense — there is some scattering.

The energy of this field theory is

$$H = \int d^3x \; \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{\lambda^2}{2}\left(\phi^2 - \frac{\mu^2}{\lambda^2}\right)^2$$

so the solution

$$\phi(x) = \frac{\mu}{\lambda} \tanh \mu(x - x_0)$$

is localized near $x = x_0$.