The determinant of A vanishes

$$0 = \det(A) = |A| = \sum_{i_1, i_2, \cdots i_n = 1}^{n} \epsilon_{i_1 i_2 \cdots i_n} a_{1 i_1} a_{2 i_2} \cdots a_{n i_n}.$$

Here $\epsilon_{12 \cdots n} = 1$ and ϵ is totally antisymmetric.

These systems of eq's. are said to be homogeneous because if $\vec{x}$ is a solution, then so is $\lambda \vec{x}$.

In n dimensions, the system

$0 = Ax$ or $0 = a_{ij} x_j$ means that $\vec{x} = (x_1, \cdots x_n)$

must be $\perp$ to every $a_i = (a_{i1}, \cdots a_{in})$. This is possible only if the n vectors

$$\vec{b}_j = (a_{1j}, a_{2j}, \cdots a_{nj}) \quad \text{satisfy}$$

$0 = \vec{b}_j x_j$. Such vectors are said to be

linearly dependent; that is, there are

coefficients x_j **not all zero** such that the sum

$$0 = \vec{b}_j x_j \iff 0 = a_{ij} x_j \quad \text{vanishes.}$$

We will see later that if the columns $\vec{b}_j$ of a matrix are linearly dependent, then the determinant

$$|A| = |b_j| = 0$$

vanishes.

A set of vectors $\vec{b}_j$ that is not linearly dependent is <u>linearly</u> <u>independent</u>. A set of n linearly independent vectors spans a space of n dimensions. So n linearly independent vectors $\vec{b}_j$ with the right coefficients x_j can make any vector $\vec{c}$: That is

$$\vec{c} = \vec{b}_j x_j$$

always has a solution if the $\vec{b}_j$ are <u>l.i.</u>

(Of course, if $c = 0$, then the unique choice is

$$x_j = 0 \quad \text{for } j = 1 - n,$$

which is another way of saying that the $\vec{b}_j$ are <u>l.i.</u>)

Inhomogeneous linear equations.

$$a_{11} x_1 + a_{12} x_2 = b_1$$
or $\qquad Ax = b$
$$a_{21} x_2 + a_{22} x_2 = b_2$$

is equivalent to $a_1 \cdot x = a_2 \cdot x = 0$ with

$$a_1 = (a_{11}, a_{12}, b_1)$$

$$a_2 = (a_{21}, a_{22}, b_2)$$ and

$$x = (x_1, x_2, -1).$$

A solution x is a vector $\perp$ to a_1 and a_2, so

$$x = \lambda \, a_1 \times a_2$$

but since $x_3 = -1 \neq 0$, we must have

$$x_3 = \lambda \, \epsilon_{3ij} \, a_{1i} \, a_{2j}$$

$$= \lambda (a_{11} a_{22} - a_{12} a_{21}) = -1 \neq 0$$

That is, the determinant of $|A|$ must <u>not</u> vanish:

$$|A| = a_{11} a_{22} - a_{12} a_{21} \neq 0.$$

We set

$$\lambda\, |A| = -1 \qquad \text{and}$$

$$\vec{x} = \lambda\, a_1 \times a_2 = -\, a_1 \times a_2 / |A|.$$

Determinants.

If $\quad A = (a_{ij})$, is a 2×2 matrix

$$\det(A) = |A| = \sum_{i,j=1}^{2} \epsilon_{ij}\, a_{1i}\, a_{2j}$$

$$= a_{11} a_{22} - a_{12} a_{21}$$

If A is 3×3, then

$$\det(A) = |A| = \sum_{ijk=1}^{3} \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}.$$

In a simpler notation, if

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \qquad \text{then}$$

$$|A| = \sum \epsilon_{ijk}\, a_i\, b_j\, c_k$$

$$= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1$$

$$+ a_3 b_1 c_2 - a_3 b_2 c_1 .$$

Laplace used minors:

$$|A| = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \sum_{j=1}^{3} (-1)^{j+1} a_j M_{1j} = \sum a_j C_{1j}$$

where C_{1j} is the cofactor

$$C_{1j} = (-1)^{j+1} M_{1j} .$$

This works well for sparce matrices,

$$D = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} = (-1)^{1+2} (1) \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= (-1)(-1)^{1+1}(-1) \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} = 1.$$

But in general, it's a long way to proceed.

If $A = (a_{ij})$ is $n \times n$, then

$$|A| = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \epsilon_{i_1 i_2 \cdots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}.$$

The antisymmetry of ϵ has many consequences.

If any two rows are equal, then the determinant vanishes. For example,

$$|A| = \sum \epsilon_{i_1 i_2 \cdots i_n} a_{1 i_1} a_{1 i_2} a_{3 i_3} \cdots a_{n i_n}$$

$$= (a_{11} a_{12} - a_{12} a_{11}) \text{ stuff } = 0.$$

Also if any two columns are equal, $|A| = 0$.

E.g.,

$$\begin{vmatrix} a_1 & a_1 \\ a_2 & a_2 \end{vmatrix} = a_1 a_2 - a_1 a_2 = 0.$$

If any two rows or columns are interchanged, the det changes sign.

$$|A'| = \sum \epsilon_{i_1 i_2 \cdots i_n} a_{2 i_1} a_{1 i_2} a_{3 i_3} \cdots a_{n i_n}$$

$$= - \sum \epsilon_{i_2 i_1 i_3 \cdots i_n} a_{1 i_2} a_{2 i_1} a_{3 i_3} \cdots a_{n i_n}$$

$$= - |A|.$$

$$|A'| = \sum \epsilon_{i_1 \cdots i_n} a_{1i_1} (c a_{2i_2}) \cdots a_{ni_n}$$

$$= c \sum \epsilon_{i_1 \cdots i_n} a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

$$= c|A|.$$

If

$$a'_{kj} = a_{kj} + c a_{\ell j},$$

then

$$|A'| = \sum \epsilon_{i_1 \cdots i_n} a_{1i_1} (a_{k i_k} + c a_{\ell i_k}) \cdots a_{ni_n}$$

$$= |A| + \sum \epsilon_{i_1 \, i_k \, i_\ell \, i_n} a_{1i_1} a_{\ell i_\ell} \cdots a_{\ell i_k} \cdots a_{ni_n}$$

$$= |A|.$$

So one may add multiples of rows (or columns) to other rows (or columns) without changing $|A|$.

So if any two rows are proportional, then $|A| = 0$. (or columns).

Suppose A is 3×3 and we want to solve $Ax=0$ with $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$. Then

$$x_1|A| = x_1\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 & a_2 & a_3 \\ b_1x_1 & b_2 & b_3 \\ c_1x_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_2 & a_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_2 & b_3 \\ c_1x_1 + c_2x_2 + c_3x_3 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix} = 0. \qquad \text{So } |A|=0.$$

But if $Ax=d$, then we get

$$x_1|A| = \begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{vmatrix} \qquad \text{so}$$

$$x_1 = \begin{vmatrix} d_1 & a_2 & a_3 \\ d_2 & b_2 & b_3 \\ d_3 & c_2 & c_3 \end{vmatrix} \Big/ |A|.$$

So the inhomogeneous equation $Ax=d$ works only if $|A|\neq 0$.

Gaussian elimination

bigger numbers

$$4x + 2y + z = 48$$

$$2x + 2y + z = 36$$

$$x + y + z = 6$$

$$x + \tfrac{1}{2}y + \tfrac{1}{4}z = 12$$

$$x + y + \tfrac{1}{2}z = 18$$

$$x + y + z = 6$$

subtract to eliminate x's

$$x + \tfrac{1}{2}y + \tfrac{1}{4}z = 12$$

$$\tfrac{1}{2}y + \tfrac{1}{4}z = 6$$

$$\tfrac{1}{2}y + \tfrac{3}{4}z = -6$$

normalize

$$x + \tfrac{1}{2}y + \tfrac{1}{4}z = 12$$

$$y + \tfrac{1}{2}z = 12$$

$$y + \tfrac{3}{2}z = -12$$

subtract $x + \frac{1}{2}y + \frac{1}{4}z = 12$

$$y + \frac{1}{2}z = 12$$

$$z = -24$$

solve for y

$$y = 12 - \frac{1}{2}z = 12 + 12 = 24$$

$$x = 12 - \frac{y}{2} - \frac{z}{4} = 12 - 12 + 6 = 6$$

check:

$$x + y + z = 6 + 24 - 24 = 6$$

$$2x + 2y + z = 12 + 48 - 24 = 36$$

$$4x + 2y + z = 24 + 48 - 24 = 48.$$

Matrices

Addition: If A and B are $n \times m$ matrices, then their sum $C = A + B$ is

$$C_{ij} = A_{ij} + B_{ij}.$$

Scalar multiplication: If A is $n \times m$, then $C = \alpha A$ is $n \times m$ and

$$C_{ij} = \alpha A_{ij}.$$

Matrix multiplication: If A is $n \times m$ and B is $m \times p$, then their product

$$C = A \; B$$

is $n \times p$ and has entries

$$C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}.$$

$$\begin{pmatrix} a & a & a & a \\ & & & \end{pmatrix} \begin{pmatrix} & & b \\ & & b \\ & & b \\ & & b \end{pmatrix} = \begin{pmatrix} & & 4ab \\ & & \end{pmatrix}$$

Transposition:

$$A^T_{ij} = A_{ji} \qquad \tilde{A}_{ij} = A_{ji}$$

$$A'_{ij} = A_{ji}$$

Complex numbers: if x and y are real, then

$$z = x + iy \quad \& \quad \bar{z} = z^* = x - iy$$

are complex. $\bar{z} = z^*$ is the c.c.

$i^2 = -1$ so $\quad z\bar{z} = x^2 + y^2 = |z|^2.$

$$A^*_{ij} = \bar{A}_{ij}$$

hermitian conjugation:

$$A^\dagger_{ij} = A^{T*}_{ij} = A^*_{ji}.$$

Equality: $A = B \iff A_{ij} = B_{ij}$ and both must be $n \times m$.

Associative: $A(BC) = (AB)C$ because

$$[A(BC)]_{ij} = \sum_k A_{ik}\left(\sum_\ell B_{k\ell} C_{\ell j}\right) = \sum_\ell \left(\sum_k A_{ik} B_{k\ell}\right) C_{\ell j}$$

$$= [(AB)C]_{ij}$$

Distributive law:

$A(B+C) = AB + AC$ because

$$[A(B+C)]_{ij} = \sum_k A_{ik}(B_{kj} + C_{kj})$$

$$= \sum_k A_{ik} B_{kj} + \sum_k A_{ik} C_{kj} = AB + AC.$$

Zero: $O_{ij} = 0$.

The set of all $n \times n$ matrices forms a ring.

But this set is not a field because $AB \neq BA$ in general and also

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $A^2 = 0$, whereas in

a field if $AB = 0$ then A or B must be 0.
A has no inverse. For if B were its inverse, then

$1 = AB \iff (AB)_{ij} = \delta_{i1}(\delta_{j2} b_{21} + \delta_{j1} b_{21})$

yet $1_{22} = 1$.

$(AB)^T = B^T A^T$ because

$$(\widetilde{AB})_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k \tilde{A}_{kj} \tilde{B}_{ik}$$

$$= \sum_k \tilde{B}_{ik} \tilde{A}_{kj} = (\tilde{B}\tilde{A})_{ij}$$

It follows that

$(AB)^\dagger = B^\dagger A^\dagger$.

If A is $n \times n$ and $|A| \neq 0$, then one may show that A has a unique inverse A^{-1} such that

$$AA^{-1} = A^{-1}A = 1 \quad \text{ie}$$

$$\sum_k A_{ik} A^{-1}_{kj} = \sum_k A^{-1}_{ik} A_{kj} = \delta_{ij}.$$

If A & B have inverses A^{-1} and B^{-1}, then $(AB)^{-1} = B^{-1}A^{-1}$ because

$AB(B^{-1}A^{-1}) = AA^{-1} = 1$ and $B^{-1}A^{-1}AB = B^{-1}B = 1$.

The most general form of a determinant is

$$\epsilon_{i_1 i_2 \cdots i_n} |A| = \Sigma A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_n j_n} \epsilon_{j_1 \cdots j_n}$$

For instance,

$$\epsilon_{21} \left| A \right| = A_{2j_1} A_{1j_2} \epsilon_{j_1 j_2}$$

$$= A_{21} A_{12} \epsilon_{12} + A_{22} A_{11} \epsilon_{21}$$

$$= A_{21} A_{12} - A_{11} A_{22} = -|A| = \epsilon_{21} |A|.$$

So

$$|AB| = \epsilon_{i_1 \cdots i_n} (AB)_{1 i_1} \cdots (AB)_{n i_n}$$

$$= \epsilon_{i_1 \cdots i_n} A_{1k_1} B_{k_1 i_1} A_{2k_2} B_{k_2 i_2} \cdots A_{nk_n} B_{k_n i_n}$$

$$= A_{1k_1} \cdots A_{nk_n} \epsilon_{k_1 \cdots k_n} |B|$$

$$= |A| \, |B|.$$

Since $|AA^{-1}| = |1| = 1 = |A| |A^{-1}|$,
it follows that

$$|A^{-1}| = 1/|A|.$$

Direct Product

$A \otimes B = C$

$C_{i_1 i_2; j_1 j_2} = A_{i_1 j_1} \; B_{i_2 j_2}$

The most common case is the

Dyadic: If A is an n-vector and B is an m-vector, then

$$\tilde{A} \cdot B = (A_1 \cdots A_n) \begin{pmatrix} B_1 \\ \\ B_m \end{pmatrix} = A \cdot B \text{ only if } n = m,$$

but even if $m \neq n$ we have

$$B \otimes \tilde{A} = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix} (A_1 \cdots A_n)$$

$$= \begin{pmatrix} B_1 A_1 & B_1 A_2 & & B_1 A_n \\ & & & \\ B_m A_1 & & & B_m A_n \end{pmatrix}$$

2 x 2 case

$$A \times B = \begin{pmatrix} A_{11}B & A_{12}B \\ A_{21}B & A_{22}B \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}$$

column

$$(A \times B)_{i_1 i_2 ; j_1 j_2} = A_{i_1 j_1} B_{i_2 j_2}$$

row

Most matrices do not commute:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is undefined.

Even if both are 2 x 2

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta d & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix}$$

Not only are the two products different; in general, they have no common elements.

Diagonal: If $A_{ij} = a_i \delta_{ij}$, then A is diagonal.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$ is diagonal.

Diagonal matrices do commute

If $A_{ij} = a_i \delta_{ij}$ and $B_{jk} = b_j \delta_{jk}$, then

$$(AB)_{ik} = \sum_j a_i \delta_{ij} b_j \delta_{jk} = \delta_{ik} a_i b_k = \delta_{ik} a_i b_i$$

$$(BA)_{ik} = b_i \delta_{ij} a_j \delta_{jk} = \delta_{ik} b_i a_k = \delta_{ik} a_i b_i.$$

Trace: If A is $n \times n$, then

$$\operatorname{tr} A = \operatorname{Tr} A = \sum_{i=1}^{n} A_{ii}.$$

$$\operatorname{tr} ABC = A_{ij} B_{jk} C_{ki}$$

$$= C_{ki} A_{ij} B_{jk} = \operatorname{tr} CAB$$

$$= B_{jk} C_{ki} A_{ij} = \operatorname{tr} BCA.$$

The trace is cyclicly invariant.

$$\operatorname{tr} AB = \operatorname{tr} BA.$$

Matrix Inversion

$$A^{-1}_{ij} = \frac{C_{ji}}{|A|} \quad \text{where} \quad C_{ij} = (-1)^{i+j} M_{ij}.$$

is the wrong way to compute inverses.

$$M_L A = A'$$

$$A'_{ij} = k a_{ij} = M_{Lil} A_{lj}$$

$$M_L A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ kA_{21} & kA_{22} & kA_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

in general to scale row i by k

$$M_{Lnm} = \delta_{nm} + \delta_{ni} \delta_{mi} (k-1).$$

$$M_L A = \begin{pmatrix} 1 & k & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+kd & b+ke & c+kf \\ d & e & f \\ g & h & i \end{pmatrix}$$

So
$$M_{Lnm} = \delta_{nm} + \delta_{nr} \delta_{mc} k$$

will add a multiple of row c to row r.

$$M_c A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix}$$

So

$$M_{c\,nm} = \delta_{nm} + \delta_{m r_2}\delta_{m r_1}$$

$$+ \delta_{n r_1}\delta_{m r_2} - \delta_{n r_1}\delta_{m r_1} - \delta_{n r_2}\delta_{m r_2}$$

will interchange rows r_1 and r_2.

Example 3.2.1 GAUSS–JORDAN MATRIX INVERSION

We want to invert the matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}. \qquad (3.53)$$

For convenience we write A and 1 side by side and carry out the identical operations on each:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.54)$$

To be systematic, we multiply each row to get $a_{k1} = 1$,

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (3.55)$$

Subtracting the first row from the second and third, we obtain

$$\begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{11}{3} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{1}{2} & 0 \\ -\frac{1}{3} & 0 & 1 \end{pmatrix}. \qquad (3.56)$$

Then we divide the second row (of **both** matrices) by $\frac{5}{6}$ and subtract $\frac{2}{3}$ times it from the first row, and $\frac{1}{3}$ times it from the third row. The results for both matrices are

$$\begin{pmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & \frac{18}{5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{3}{5} & -\frac{2}{5} & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \\ -\frac{1}{5} & -\frac{1}{5} & 1 \end{pmatrix}. \tag{3.57}$$

We divide the third row (of **both** matrices) by $\frac{18}{5}$. Then as the last step $\frac{1}{5}$ times the third row is subtracted from each of the first two rows (of both matrices). Our final pair is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{A}^{-1} = \begin{pmatrix} \frac{11}{18} & -\frac{7}{18} & -\frac{1}{18} \\ -\frac{7}{18} & \frac{11}{18} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} \end{pmatrix}. \tag{3.58}$$

The check is to multiply the original A by the calculated A^{-1} to see if we really do get the unit matrix 1.

As with the Gauss–Jordan solution of simultaneous linear algebraic equations, this technique is well adapted to computers. Indeed, this Gauss–Jordan matrix inversion technique will probably be available in the program library as a subroutine (see Sections 2.3 and 2.4 of Press *et al.*, loc. cit.). ■

Dirac's notation:

We write the orthonormal basis vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \cdots \qquad e_n = \begin{pmatrix} 0 \\ 0 \\ \\ 1 \end{pmatrix}$$

as $|1\rangle$ $|2\rangle$ $|n\rangle$.

Their transposes

$\tilde{e}_1 = (1\,0 \quad 0)$ $\quad \cdots \quad$ $\tilde{e}_n = (0 \quad 0\ 1)$

we write as

$\langle 1|$ $\quad \langle 2|$ $\quad \langle n|$.

Then we write

$$\tilde{e}_n \cdot e_m = \langle n|m\rangle = \delta_{nm}.$$

An arbitrary vector $|\psi\rangle$ in C^n is

$$|\psi\rangle = \sum_{i=1}^{n} \psi_i e_i = \sum_{i=1}^{n} \psi_i |i\rangle = \psi_i |i\rangle$$

and its hermitian adjoint is

$$\langle\psi| = \sum \psi_i^* \tilde{e}_i = \sum \psi_i^* e_i^\dagger = \sum \psi_i^* \langle i|$$

$$= \psi_i^* \langle i|.$$

Now the inner product of two vectors ψ and ϕ is

$$(\psi, \phi) = \langle\psi|\phi\rangle = \sum_{i=1}^{n} \psi_i^* e_i^\dagger \sum_{j=1}^{n} \phi_j e_j$$

$$= \sum_{i,j} \psi_i^* \langle i|j\rangle \phi_j = \sum_i \psi_i^* \phi_i.$$

$$= \langle\phi|\psi\rangle^* = \left[\sum \phi_i^* \langle i|j\rangle \psi_j\right]^* = \left(\sum \phi_i^* \psi_i\right)^*.$$

But we must also consider outer products

$$\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}.$$

In Dirac's notation, we write such things as

$$|n\rangle\langle m| = e_n e_m^{+} = |n\rangle \otimes \langle m|$$

where we have used the direct-product notation of Eq. (3.45). Frankly 'outer product' is the more commonly used term.

Now there are infinitely many sets of ON bases,

$$\langle n|m\rangle = \delta_{nm} \quad \& \quad \langle a|b\rangle = \delta_{ab} \quad \text{etc.}$$

In this general context, $\langle n|$ means the hermitian adjoint, not just the transpose, since these vectors are complex in general.

Each of these bases is complete

$$1 = \sum |n\rangle\langle n| = \sum |a\rangle\langle a| \quad , \text{etc.,}$$

as well as ON $\langle n|m\rangle = \delta_{nm}$ etc.

Change of basis:

$$|n\rangle = \sum_a |a\rangle\langle a|n\rangle$$

So the matrix that takes us from $|a\rangle$ to $|n\rangle$ is $\langle a|n\rangle$. Since both bases are ON,

$$\delta_{mn} = \langle m|n\rangle = \sum_a \langle m|a\rangle\langle a|n\rangle .$$

In matrix notation, if

$$U_{an} = \langle a|n\rangle \text{, then } U^*_{am} = \langle m|a\rangle,\ U^\dagger_{ma} = \langle m|a\rangle$$

$$\delta_{mn} = \sum_a \langle m|a\rangle U_{an} = \sum U^*_{am} U_{an}$$

$$= \sum U^\dagger_{ma} U_{an} = \langle m|a\rangle\langle a|n\rangle = \delta_{mn} .$$

So the matrix U_{an} is unitary

$$1 = U^\dagger U .$$

$$|a\rangle = \sum |n\rangle\langle n|a\rangle$$

So U^* takes $|m\rangle$ to $|a\rangle$ and

$$\langle b|a\rangle = \delta_{ba} = \sum \langle b|n\rangle\langle n|a\rangle = \sum_n U_{bn} U^\dagger_{na} ,$$

$$1 = U U^\dagger .$$

So unitary matrices shift us from one CONB to another.

If the vectors are <u>all</u> <u>real</u>, then none of the complex conjugation was necessary and we have

$$1 = \tilde{u}\, u = u\, \tilde{u}$$

as well as $\langle a|n\rangle^* = \langle a|n\rangle = \langle n|a\rangle$.

In this case the matrices are orthogonal

$$1 = \tilde{O}\, O = O\, \tilde{O}.$$

Changed coefficients:

$$|\psi\rangle = \sum |n\rangle\langle n|\psi\rangle$$

$$= \sum |a\rangle\langle a|\psi\rangle$$

then

$$\langle n|\psi\rangle = \sum_a \langle n|a\rangle\langle a|\psi\rangle = \sum u^*_{an}\, \psi_a$$

$$= \sum u^+_{na}\, \psi_a$$

Again

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$$

$$= \sum_n |n\rangle \sum_{a'} \langle n|a'\rangle\langle a'|\psi\rangle = \sum_a |a\rangle\langle a| \sum_n |n\rangle\langle n| \sum_{a'} |a'\rangle\langle a'|\psi\rangle$$

$$= \sum_a |a\rangle \sum_n \langle a|n\rangle\langle n| \sum_{a'} |a'\rangle\langle a'|\psi\rangle$$

$$= \sum_{a,a',n} |a\rangle\, U_{an} U^*_{a'n} \langle a'|\psi\rangle$$

$$= \sum_{aa'n} |a\rangle\, U_{an} U^\dagger_{na'} \langle a'|\psi\rangle$$

$$= \sum_{a,a'} |a\rangle\, (UU^\dagger)_{aa'} \langle a'|\psi\rangle$$

$$= \sum_{aa'} |a\rangle\, \delta_{aa'} \langle a'|\psi\rangle = \sum |a\rangle\langle a|\psi\rangle.$$

$e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$ is a unitary operator that implements rotations by angle $\theta = |\vec{\theta}|$ about axis $\hat{\theta}$ in the right-handed active sense.

Why unitary? Because it's the exponential of a hermitian matrix.

$$U = e^{iH}$$

where $H^{+} = H$. So

$$U^{+}U = e^{-iH^{+}} e^{iH} = e^{-iH} e^{iH} = 1$$

since $[H, H] = 0$. Why is $\sigma \cdot \theta$ hermitian? Because $\vec{\theta}$ is a real vector and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tilde{\sigma}_1^{*}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \tilde{\sigma}_2^{*}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3^{\dagger}.$$

Now

$$\sigma_i \sigma_j = \delta_{ij} + i\sum_{k=1}^{3} \epsilon_{ijk}\,\sigma_k \qquad \left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\sum \epsilon_{ijk}\frac{\sigma_k}{2}$$

e.g. $\quad \sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{11}$

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\epsilon_{123}\,\sigma_3$$

So

$$(\theta\cdot\sigma)^2 = \sum_{ij=1}^{3} \theta_i\theta_j\,\sigma_i\sigma_j = \sum \underset{\substack{\uparrow \\ S}}{\theta_i\theta_j}\big(\delta_{ij} + i\underset{\substack{\uparrow \\ A.S}}{\epsilon_{ijk}}\,\sigma_k\big)$$

$$= \sum_i \theta_i^2 = \vec{\theta}^{\,2}.$$

So

$$e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = \sum_{n=0}^{\infty}\frac{\left(-i\theta\cdot\frac{\sigma}{2}\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{\left(-i\theta\cdot\frac{\sigma}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \sum \frac{(-1)^n\left(\frac{\theta}{2}\right)^{2n}}{(2n)!} - i\,\hat{\theta}\cdot\sigma\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{\theta}{2}\right)^{2n+1}}{(2n+1)!}$$

$$= \cos\frac{\theta}{2} - i\,\hat{\theta}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}.$$

$$e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{\theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\hat{\theta}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\frac{\theta}{2}$$

$$-i\hat{\theta}_2\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\frac{\theta}{2} - i\hat{\theta}_3\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= \cos\frac{\theta}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\sin\frac{\theta}{2}\begin{pmatrix}\hat{\theta}_3 & \hat{\theta}_1 - i\hat{\theta}_2\\ \hat{\theta}_1 + i\hat{\theta}_2 & -\hat{\theta}_3\end{pmatrix}.$$

If $\vec{\mathcal{J}} = \frac{\hbar\vec{\sigma}}{2}$, then

$$U(\theta) = e^{-i\vec{\theta}\cdot\vec{\mathcal{J}}/\hbar} = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}.$$

The 3×3 rotation group is generated by

$$J^k_{ij} = i\epsilon_{ikj} \qquad (M^k \text{ in book})$$

and

$$\vec{\mathcal{J}} = \hbar\vec{J}.$$

The q.-m. operator is

$$U(\theta) = e^{-i\vec{\theta}\cdot\vec{\mathcal{J}}/\hbar} = e^{-i\vec{\theta}\cdot\vec{J}}.$$

One may show that

$$e^{-i\vec{\theta}\cdot\vec{J}} = \cos\theta - i\hat{\theta}\cdot A\sin\theta + (1-\cos\theta)\,\hat{\theta}\,\hat{\theta}^T$$

or

$$\left(e^{-i\vec{\theta}\cdot\vec{J}}\right)_{ij} = U(\vec{\theta})_{ij} = \delta_{ij}\cos\theta - \sin\theta\,\epsilon_{ijk}\hat{\theta}_k + (1-\cos\theta)\,\hat{\theta}_i\,\hat{\theta}_j \quad ,$$

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This is the ordinary rotation matrix.

Often one diagonalizes J^3 by a unitary matrix obtaining

$$\vec{J}' = u^+ \vec{J} u$$

$$J_1' = \frac{1}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix} \qquad J_2' = \frac{1}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}$$

$$J_3' = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} ,$$

as in Ex. 3.2.15.

These satisfy

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad \text{and}$$

$$U^\dagger [J_i, J_j] U = [J_i', J_j'] = i\epsilon_{ijk} U^\dagger J_k U$$

$$= i\epsilon_{ijk} J_k'.$$

So the commutators of a Lie algebra are invariant under a unitary transformation.

The unitary operator U that takes us from one O.N. basis $\{|\alpha_n\rangle\}$ to another ON basis $\{|\beta_n\rangle\}$ is

$$U = \sum_{n=1}^{N} |\beta_n\rangle\langle\alpha_n| ,$$

since

$$U|\alpha_m\rangle = \sum |\beta_n\rangle\langle\alpha_n|\alpha_m\rangle$$

$$= \sum |\beta_n\rangle \delta_{nm} = |\beta_m\rangle.$$

$$U^\dagger = \sum |\alpha_n\rangle\langle\beta_n| \qquad \text{so}$$

$$UU^\dagger = \sum |\beta_n\rangle\langle\alpha_n| \sum |\alpha_m\rangle\langle\beta_m|$$

$$= \sum |\beta_n\rangle \delta_{nm} \langle\beta_m| = \sum_n |\beta_n\rangle\langle\beta_n|$$

$$= 1$$

$$U^\dagger U = \sum |\alpha_n\rangle\langle\beta_n| \sum |\beta_m\rangle\langle\alpha_m|$$

$$= \sum |\alpha_n\rangle \delta_{nm} \langle\alpha_m|$$

$$= \sum_n |\alpha_n\rangle\langle\alpha_n| = 1.$$

Euler angles $R(\vec{\theta}) = e^{-i\theta\cdot J/\hbar}$

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

$$R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)$$

$$R_{z'}(\gamma) = R_{y'}(\beta) \; R_z(\gamma) R_{y'}^{-1}(\beta) \qquad \text{so}$$

$$R(\alpha,\beta,\gamma) = R_{y'}(\beta) \; R_z(\gamma) R_{y'}^{-1}(\beta) R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\alpha)$$

$$= R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha) R_z(\gamma) \; R_z(\alpha) R_y^{-1}(\beta) R_z^{-1}(\alpha) R_z(\alpha) R_y(\beta)$$

$$= R_z(\alpha) \; R_y(\beta) \; R_z(\gamma) \; R_y^{-1}(\beta) R_y(\beta)$$

$$= \; R_z(\alpha) \; R_y(\beta) \; R_z(\gamma),$$

which is (3.93) with α and γ interchanged. We use $R_y(\beta)$ rather than $R_x(\beta)$ because at least for $j = 1/2$ $R_y(\beta)$ is real.

$$\mathcal{D}^{1/2}_{m'm}(\alpha\beta\gamma) = \langle j=\tfrac{1}{2}, m' | \, e^{-iJ_z\alpha/\hbar} \; e^{-iJ_y\beta/\hbar} \; e^{-iJ_z\gamma/\hbar} \, | j=\tfrac{1}{2}, m\rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \langle \tfrac{1}{2} m' | \, e^{-iJ_y\beta/\hbar} \, | \tfrac{1}{2} m\rangle$$

$$= e^{-i(m'\alpha + m\gamma)} \left(e^{-i\sigma_y\beta/2} \right)_{m'm}$$

$$= e^{-i(m'\alpha + m\gamma)} \left(\cos\beta/2 - i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sin\beta/2 \right)_{m'm}$$

$$\mathcal{D}^{1/2}_{m'm}(\alpha\beta\gamma) = e^{-i(m'\alpha + m\gamma)} \left[\delta_{m'm} \cos\beta/2 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{m'm} \sin\beta/2 \right]$$

Any matrix A can be written as

$$A = \frac{1}{2}(A + A^\dagger) + \frac{1}{2}(A - A^\dagger)$$

hermitian A.H.

The Dirac matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$ 4×4 in $4D$.

Eigenvectors

$$A|x\rangle = \lambda|x\rangle$$

$$(A-\lambda I)|x\rangle = 0 \iff (A-\lambda I)_{ij}\, x_j = 0$$

columns of $A-\lambda I$ are linearly dependent so

S. $\det(A-\lambda I) = 0.$

If A is $n\times n$, then this is an algebraic equation of the form

$$(-\lambda)^n + \cdots + \det A = 0.$$ characteristic polynomial

It has n roots, in general complex.

If $A = A^\dagger$, then one may say more:

$$A|n\rangle = a_n|n\rangle$$

$$A|m\rangle = a_m|m\rangle$$

$$\langle m|A|n\rangle = a_n\langle m|n\rangle$$

cc this eq: $\langle n|A^\dagger|m\rangle = \langle n|A|m\rangle = a_n^*\langle n|m\rangle$

but

$$\langle n|A|m\rangle = a_m\langle n|m\rangle.$$

So

$$0 = (a_m - a_n^*)\langle n|m\rangle.$$

Set $n = m$:

$$0 = (a_m - a_n^*) \langle n | n \rangle$$

So the e.v.'s are real, $a_n = a_n^*$.

Then

$$0 = (a_m - a_n) \langle n | m \rangle .$$

Then $\langle n | m \rangle = 0$, unless $a_m = a_n$.

So non-degenerate e.v.'s are $\perp$.

So an $n \times n$ hermitian matrix with n distinct e.v.'s, all different, has n ON ~~states~~ $|n\rangle$

$$A | n \rangle = a_n | n \rangle$$

$$\langle n | m \rangle = \delta_{nm} .$$

These n ON vectors obviously are complete in this n-dimensional space.

Imagine general case in which two or more e.v.'s are equal. In this case, add a tiny perturbation V,

$$V^\dagger = V ,$$

so as to break ~~all~~ degeneracies. Then

$$(A + \epsilon V)\,|n,\epsilon\rangle = a_n(\epsilon)\,|n,\epsilon\rangle$$

and $$\langle n,\epsilon | m,\epsilon\rangle = \delta_{nm}$$

because all $a_n(\epsilon)$ are different. Now let $\epsilon \to 0$. The e.v.'s $|n,\epsilon\rangle$ remain $\perp$, and so they remain complete.

Thus every $n \times n$ hermitian matrix $A = A^\dagger$ has n ON complete ev's $|n\rangle$

$$A|n\rangle = a_n|n\rangle$$

$$I = \sum |n\rangle\langle n|$$

$$\langle n|m\rangle = \delta_{nm}.$$

Now consider the product P

$$P = \prod_{n=1}^{N} (A - a_n)$$

which is independent of the order of the factors. So

$$P|n\rangle = \Big[\prod_{i\neq n} (A - a_i)\Big](A - a_n)|n\rangle = 0.$$

So $$P\,I = P \sum |n\rangle\langle n| = 0.$$

So $$P = 0.$$

Cayley & Hamilton ~~showed~~ that if

$$0 = \det(A - \lambda I) = (-\lambda)^n + \cdots + \det A$$

$$= b_0 + b_1 \lambda + \cdots + (-1)^n \lambda^n$$

then

$$0 = b_0 I + b_1 A + \cdots + (-1)^n A^n.$$

Every square matrix satisfies its characteristic equation.

Normal Matrices

$$[A, A^\dagger] = 0$$

Let $B_n = A - a_n I$.

$$A|n\rangle = a_n|n\rangle$$

$$[B_n, B_n^\dagger] = [A - a_n I, A^\dagger - a_n^* I] = [A, A^\dagger] = 0$$

If $A|n\rangle = a_n|n\rangle$, then $B_n|n\rangle = 0$. So

$$\langle n|B^\dagger B|n\rangle = \langle n|B B^\dagger|n\rangle = 0 \quad \text{so}$$

$$0 = B^\dagger|n\rangle \quad \text{or} \quad A^\dagger|n\rangle = a_n^*|n\rangle.$$

Say $$A|n\rangle = a_n|n\rangle$$

$$A|m\rangle = a_m|m\rangle.$$

Then

$$\langle n|A|m\rangle = a_m \langle n|m\rangle$$ while

$$\langle n|A = (A^\dagger|n\rangle)^\dagger = (a_n^*|n\rangle)^\dagger = a_n\langle n|,$$ so

$$\langle n|A|m\rangle = a_n\langle n|m\rangle.$$ So

$$0 = (a_n - a_m)\langle n|m\rangle,$$ So the e.v.'s of

different e.v.'s are $\perp$.

Let $A(\epsilon) = A + \epsilon V$ be normal where

V breaks any degeneracy. Then if A is

$N \times N$, all N ev's $a_n(\epsilon)$ will be distinct,

the N ev's $|n,\epsilon\rangle$ will be $\perp$ and complete

$$I = \sum_{n=1}^{N} |n,\epsilon\rangle\langle n,\epsilon|.$$

Now let $\epsilon \to 0$. A has N CON ev's

$|n\rangle$.

If A is $N \times N$ but not necessarily normal, then it has N rev's an

$A|n\rangle = a_n|n\rangle$ which are the roots of

$0 = \det(A - \lambda I)$.

Similarly, it will have N lev's

$\langle i | A = a_i \langle i|$

where the N ev's a_i are the roots of the same eq.

$$0 = \det(A - \lambda I).$$

It gets better: Let A be $N \times M$ with complex elements perhaps. Then $\exists$ unitary matrices U and $V \ni$

$$A = U S V$$

where U is $N \times N$, S is $N \times M$, and V is $M \times M$. S is zero except for non-negative singular values $s_i \geq 0$

for $i = 1 \ldots \min(N, M)$. This is the singular-value decomposition. Note that the matrix A is a perfectly general matrix with N rows and M columns.

In Dirac notation

$$V = \sum_{n=1}^{M} |e_n\rangle\langle v_n|$$

takes us from the rsv's $|v_n\rangle$ to the standard ON vectors $|e_n\rangle_i = \begin{pmatrix} \delta_{in} \end{pmatrix}$.

Suppose $M > N$. Then, e.g.

$$S = \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{pmatrix}$$

and so $\quad SV|3\rangle = 0$ while

$$SV|v_n\rangle = s_n |e_n\rangle \quad \text{for } n = 1, 2.$$

$$U = \sum_{i=1}^{N} |\ell_i\rangle\langle e_i|$$

So

$$A|r,n\rangle = USV|r,n\rangle = U s_n|e,n\rangle = s_n|l,n\rangle$$

for $n = 1,2$ $\quad\quad A|r,3\rangle = 0$

So

$$A = USV$$

$$= \underbrace{\sum_{i=1}^{N} |l,i\rangle\langle l,e,i|}_{N \text{ vectors}} \underbrace{\sum_{j=1}^{\min(N,M)} s_j |l,e,j\rangle\langle r,e,j|}_{N \times M \text{ vectors}} \underbrace{\sum_{k=1}^{M} |r,e,k\rangle\langle r,k|}_{M \text{ vectors}}$$

Condition number

$$A\,|x\rangle = |y\rangle \qquad \text{if } |A| \neq 0, \text{ then}$$

$$|x\rangle = A^{-1}|y\rangle$$

Now change $|y\rangle$ by $|\delta y\rangle$, then

$$|\delta x\rangle = A^{-1}\,|\delta y\rangle$$

$$\left(\frac{\langle \delta x | \delta x\rangle}{\langle x | x\rangle}\right)^{\frac{1}{2}} \leq K(A) \left[\frac{\langle \delta y | \delta y\rangle}{\langle y | y\rangle}\right]^{1/2}.$$

If $A = A^{\dagger}$ then

$$K(A) = \frac{|\lambda|_{max}}{|\lambda|_{min}}$$

where $\quad A\,|n\rangle = \lambda_n |n\rangle.$

Turing's estimate for general A is

$$K(A) = n \max(A_{ij}) \max(A^{-1}_{ij}).$$

$$H_{ij} = (i+j-1)^{-1} \quad \text{is ill conditioned.}$$

General mechanical system

$$m \ddot{x}_i = - \frac{\partial V(x_1 \cdots x_N)}{\partial x_i} .$$

For small vibrations about the equilibrium position

$$0 = \frac{\partial V}{\partial x_i}(x_1^0 \cdots x_N^0) .$$

Then

$$V(x_1 \cdots x_N) = V(x^0) + \sum_{i,j=1}^{N} \frac{1}{2!} \frac{\partial^2}{\partial x_i \partial x_j} V(x^0) \, (x_i - x_i^0)(x_j - x_j^0)$$

eg.

$$V = (x-1)^2 + (x-1)(y-1) + (y-1)^2$$

$$V_x = 2(x-1) + (y-1) \qquad V_y = (x-1) + 2(y-1)$$

$$V_{xx} = 2 \qquad V_{xy} = 1 \qquad V_{yy} = 2$$

$$V = \frac{1}{2}\left[2(x-1)^2 + 2(x-1)(y-1) + 2(y-1)^2 \right]$$

$$= (x-1)^2 + (x-1)(y-1) + (y-1)^2 \qquad \text{OK} .$$