Let $X = g$ which is real and symmetric.
Then

$$\delta \log \det(g_{ij}) = \delta \log g = \frac{1}{g}\, \delta g$$

$$= \operatorname{tr} g^{-1} \delta g = g^{ij}\, \delta g_{ji} \quad \text{or}$$

$$\frac{\partial g}{\partial q^k} = g\, g^{ij} \frac{\partial g_{ji}}{\partial q^k}$$

So

$$\Gamma^i_{ik} = \frac{1}{2} g^{im} \frac{\partial g_{im}}{\partial q^k} = \frac{1}{2g} \frac{\partial g}{\partial q^k}$$

But

$$\frac{1}{g^{1/2}} \frac{\partial g^{1/2}}{\partial q^k} = \frac{1}{g^{\frac{1}{2}}} \frac{1}{2} \frac{1}{g^{1/2}} \frac{\partial g}{\partial q^k} = \frac{1}{2g} \frac{\partial g}{\partial q^k}$$

So

$$\nabla \cdot V = V^i_{;i} = \frac{\partial V^i}{\partial q^i} + V^k \Gamma^i_{ik}$$

$$= \frac{\partial V^k}{\partial q^k} + \frac{V^k}{2g} \frac{\partial g}{\partial q^k} = \frac{\partial V^k}{\partial q^k} + \frac{V^k}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^k}$$

So

$$\nabla \cdot V = V^i_{;i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^k} \left(V^k \sqrt{g} \right)$$

Note

$$g = \det g_{ij} = (h_1 h_2 h_3)^2$$

So

$$\sqrt{g} = h_1 h_2 h_3$$

And

$$V = V^i \epsilon_i = \overset{2.2}{V_i} e_i = V_i \epsilon^i \qquad \epsilon_i = h_i e_i$$

$$= \frac{\overset{2.2}{V_i}}{h_i} \epsilon_i = \frac{V_i}{h_i} h_i e_i = \overset{2.2}{V_i} e_i$$

So

$$\nabla \cdot V = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q^k} \left(\frac{\overset{2.2}{V_k}}{h_k} h_1 h_2 h_3 \right) \qquad (2.17)$$

and we have agreement with (2.17).

Laplacian

Set $V^k = g^{ki} \dfrac{\partial \psi}{\partial q^i}$, then

$$\nabla \cdot \nabla \psi = \frac{1}{g^{1/2}} \frac{\partial}{\partial q^k} \left(g^{1/2} g^{ki} \frac{\partial \psi}{\partial q^i} \right).$$

For orthogonal systems

$$g^{ik} = \delta_{ik} h_i^{-2} \qquad \text{whence}$$

$$\nabla \cdot \nabla \psi = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial q^k} \left(\frac{h_1 h_2 h_3}{h_k^2} \frac{\partial \psi}{\partial q^k} \right)$$

which is (2.18a).

<u>Curl</u> The covariant curl

$$V_{k;l} - V_{l;k} = \frac{\partial V_k}{\partial q^l} - \Gamma_{l\,k}^{\,m} V_m$$

$$- \left(\frac{\partial V_l}{\partial q^k} - \Gamma_{k\,l}^{\,m} V_m \right)$$

is just the old curl

$$V_{k;l} - V_{l;k} = V_{k,l} - V_{l,k} = \partial_l V_k - \partial_k V_l$$

because $\Gamma^m_{kl} = \Gamma^m_{lk}$.

The geodesic

$$\frac{d^2 x^i}{d\tau^2} = - \Gamma^i_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau}$$

in the presence of E & M becomes

$$m \frac{d^2 x^i}{d\tau^2} = e F^{\cdot i}{}_{j} \frac{dx^j}{d\tau} - m \Gamma^i_{jk} \frac{dx^j}{d\tau} \frac{dx^k}{d\tau}$$

where $F^i{}_j = g_{jk} F^{ik}$ and

$$F^{ik}{}_{;i} = - J^k$$

$$F_{mn;l} + F_{lm;n} + F_{nl;m} = 0$$

are Maxwell's equations. But because $F^{ik} = - F^{ki}$, these reduce to

$$\frac{\partial}{\partial x^m} \sqrt{g}\, F^{mn} = - \sqrt{g}\, J^n$$

and,

$$\partial_l F_{mn} + \partial_n F_{lm} + \partial_m F_{nl} = 0.$$

Newtonian Limit* $e=0$

slow motion in a weak static gravitational field:

Neglect $\frac{d\vec{x}}{d\tau}$ with respect to $\frac{dt}{d\tau}$.

$$\frac{d^2x^m}{d\tau^2} + \Gamma^m_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

Recall

$$\Gamma^m_{00} = \frac{1}{2} g^{km}\left(\underset{0}{\underset{\parallel}{\partial_0 g_{k0}}} + \underset{0}{\underset{\parallel}{\partial_0 g_{0k}}} - \partial_k g_{00}\right)$$

$$= -\frac{1}{2} g^{km}\,\partial_k g_{00}\,,$$

Weak field means

$$g_{ab} = \eta_{ab} + h_{ab}$$

$$\eta_{ab} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad |h_{ab}| \ll 1.$$

Then

$$\Gamma_{00}^{m} = -\frac{1}{2}\,\partial_m h_{00}$$

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00}$$

$$\frac{d^2\vec{x}}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \vec{\nabla} h_{00}$$

$$\frac{d^2t}{d\tau^2} = -\Gamma^0_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

since $\Gamma^0_{00} = -\frac{1}{2}\dot{g}_{00} = 0$.

So $\frac{dt}{d\tau}$ = constant.

$$\frac{d^2\vec{x}}{dt^2} = \frac{1}{\left(\frac{dt}{d\tau}\right)^2}\frac{d^2x}{d\tau^2} = \frac{1}{2}\vec{\nabla} h_{00} = -\nabla\phi$$

$$\phi = -\frac{GM}{r}$$

So $h_{00} = -2\phi$ + constant $= -2\phi$.

$$g_{00} = -1 - 2\phi$$

Here $c=1$ and ϕ is dimensionless.

$\phi = 10^{-39}$ surface of proton

$\phi = 10^{-9}$ earth

$\phi = 10^{-6}$ sun

$\phi = 10^{-4}$ white dwarf

In cgs $[\phi] = \ell^2/t^2$.

Clock moving

$$\Delta\tau = (-ds^2)^{1/2}$$

In flat space

$$\Delta\tau = (-\eta_{\mu\nu}\, dx^\mu dx^\nu)^{1/2}$$

$$= (c^2 dt^2 - v^2 dt^2)^{1/2}$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{1/2} c\, dt$$

So the proper time is slowed down.

Muons decay in 2.2 μs.

Even at c a muon would take

$$t = \frac{10\text{km}}{300{,}000 \text{ km/s}} = \frac{1}{30{,}000} \text{ sec}$$

$$= 3 \times 10^{-5} \text{ s.}$$

So most muons would be

$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ before hitting

earth. But if $v = .99c$, then

the proper time is

$$\Delta \tau = (1-(.99)^2)^{1/2}\, dt$$

$$= (1-.9801)^{1/2}\, dt = 0.14\, dt$$

and so

$$\Delta t = 0.14 \times 3 \times 10^{-5} = 4\ \mu s$$

So about 10% would survive.

e^{-2} vs e^{-15}.

How about a clock in a gravitational field?. $g_{\mu\nu} = E_\mu^a \eta_{ab} E_\nu^b$

$$c\,d\tau = (-g_{\mu\nu}\,dx^\mu dx^\nu)^{1/2}$$

If clock is at rest, then

$$c\,d\tau = (-g_{00})^{1/2}\,c\,dt$$

So
$$d\tau = (1+2\phi)^{1/2}\,dt$$

$$d\tau = \left(1 - 2\frac{GM}{r}\right)^{1/2}dt$$

So clocks slow down near big masses.

So
$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} = \sqrt{\frac{-1-2\phi_2}{-1-2\phi_1}}$$

$$\frac{\nu_1+\nu_2-\nu_1}{\nu_1} = \frac{1+\phi_2}{1+\phi_1} = 1+\phi_2-\phi_1 = 1+\frac{\Delta\nu}{\nu}$$

$$\frac{\Delta\nu}{\nu} = \Delta\phi \qquad \text{Since } \phi_\odot = -2.12\ 10^{-6}$$

$$\frac{\Delta\nu}{\nu} = -2.12\times10^{-6}$$ light from sun is shifted to red by 2 parts in 10^6.

Pound & Rebka 1960.

$$\Delta\phi = -2.46 \times 10^{-15} = -\frac{980 \frac{cm}{sec^2} \, 2260 \text{ cm}}{c^2}$$

HW4: 2.5.(13, 22, 23); 2.9.13; 2.10.8; 2.20.11, 2.10.15; &
2.11.(2,3).

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} \qquad \text{geodesic}$$

or

$$m\frac{d^2x^\mu}{d\tau^2} = eF^\mu{}_\nu\frac{dx^\nu}{d\tau} - m\Gamma_{\nu\lambda}{}^\mu\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$$

$$R^\lambda_{\mu\nu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\alpha}R^\alpha{}_{\mu\nu\kappa}$$

$$R_{\mu\kappa} = g^{\lambda\nu}R_{\lambda\mu\nu\kappa} \qquad \text{Ricci}$$

$$= g^{\lambda\nu}g_{\lambda\alpha}R^\alpha{}_{\mu\nu\kappa}$$

$$R_{\mu\kappa} = \delta^\nu_\alpha R^\alpha{}_{\mu\nu\kappa} = R^\nu{}_{\mu\nu\kappa}$$

$$R = g^{\lambda\nu}g^{\mu\kappa}R_{\lambda\mu\nu\kappa} = g^{\lambda\nu}g^{\mu\kappa}g_{\lambda\alpha}R^\alpha{}_{\mu\nu\kappa}$$

$$= \delta^\nu_\alpha g^{\mu\kappa}R^\alpha{}_{\mu\nu\kappa} = g^{\mu\kappa}R^\nu_{\mu\nu\kappa}$$

$$= g^{\mu\kappa}R_{\mu\kappa}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \xleftarrow{-\lambda g_{\mu\nu}} = -8\pi G T_{\mu\nu}$$

8.3

Some quick applications to E & M and QM:

Laplace's eqn

$$\nabla^2\phi = \Delta\phi = 0$$

$$\nabla^2\begin{pmatrix}E\\B\end{pmatrix} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\begin{pmatrix}E\\B\end{pmatrix}\left(\begin{array}{l}\text{if monochromatic}\\ = -\frac{\omega^2}{c^2}\begin{pmatrix}E\\B\end{pmatrix} = -k^2\begin{pmatrix}E\\B\end{pmatrix}\end{array}\right)$$

$$\left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r})\right]\Psi(r,t) = i\hbar\frac{\partial}{\partial t}\Psi(r,t)$$

If $\Psi = e^{-iEt/\hbar}\psi(r)$, then

$$\left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r})\right]\psi(\vec{r})e^{-iEt/\hbar} = E\psi(\vec{r})e^{-iEt/\hbar}$$

and we have

$$\left[-\frac{\hbar^2}{2m}\Delta + V\right]\psi = E\psi .$$

For the case of constant V_0 this is

$$-\Delta\psi = \frac{2m}{\hbar^2}\left[E - V_0\right]\psi = k^2\psi$$

So we often encounter equations of the form

$$\nabla^2\psi + k^2\psi = 0 .$$

(x,y,z):

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + h^2\psi = 0$$

Try $\quad \psi(x,y,z) = X(x)\,Y(y)\,Z(z)$; then

$$YZX'' + XZY'' + XYZ'' + h^2XYZ = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + h^2 = 0.$$

So

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k^2 - \frac{Y''}{Y} - \frac{Z''}{Z} = f(y,z)$$

function of x. $\qquad$ So

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\ell^2$$

$$-h^2 - \frac{Y''}{Y} - \frac{Z''}{Z} = -\ell^2$$

So

$e(y) \quad \dfrac{Y''}{Y} = \ell^2 - h^2 - \dfrac{Z''}{Z} = f(z)$

So $\qquad \dfrac{Y''(y)}{Y(y)} = -m^2$

And so

$$\frac{Z''}{Z} = l^2 - k^2 + m^2 = -n^2$$

We chose minus signs because Δ and $\frac{d^2}{dx^2}$ are negative operators.

The simplest solutions here are

$$X(x) = A e^{ilx} + B e^{-ilx}$$

or if x is real, then

$$X(x) = A_l \sin(lx) + B_l \cos(lx), \text{ so that}$$

$$X' = lA \cos lx - lB \sin lx$$

$$X''(x) = -l^2 A \sin lx - l^2 B \cos lx = -l^2 X(x).$$

So $\quad X_l(x) = A_l \sin lx + B_l \cos lx$

$$Y_m(y) = C_m \sin(my) + D_m \cos(my)$$

and

$$Z_n(z) = E_n \sin(nz) + F_n \cos(nz).$$

The general solution is:

$$\psi_k(x,y,z) = \sum_{\substack{l,m,n \\ l^2+m^2+n^2=k^2}} (A_l \sin lx + B_l \cos lx)(C_m \sin my + D_m \cos my)(E_n \sin nz + F_n \cos nz).$$

It is the linearity of the equation

$$-\Delta\psi = k^2\psi$$

that allows one to superpose solutions:

If $\quad -\Delta\psi_1 = k^2\psi_1$ and

$$-\Delta\psi_2 = k^2\psi_2, \text{ then}$$

$$-\Delta(\psi_1+\psi_2) = -\Delta\psi_1 - \Delta\psi_2 = k^2\psi_1 + k^2\psi_2$$
$$= k^2(\psi_1+\psi_2).$$

(ρ,φ,z):
$$-\Delta\psi = k^2\psi \quad \text{or} \quad (2.33)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} + k^2\psi = 0$$

Let
$$\psi(\rho,\varphi,z) = P(\rho)\,\Phi(\varphi)\,Z(z).$$

Then

$$\Phi Z \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{PZ}{\rho^2}\frac{d^2\Phi}{d\varphi^2} + P\Phi\frac{d^2Z}{dz^2} + h^2 P\Phi Z = 0$$

or

$$-\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{\rho P}\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\varphi^2} + h^2 = -\ell^2$$

then

$$Z'' = \ell^2 Z$$

or

$$Z_\ell(z) = E_\ell e^{\ell z} + F_\ell e^{-\ell z}$$

Notice that these solutions diverge as $|z| \to \infty$.

So now

$$\frac{1}{\rho P}(\rho P')' + \frac{1}{\rho^2\Phi}\Phi'' + h^2 = -\ell^2$$

let $\quad h^2 + \ell^2 = n^2$. Then

$$g(\rho) = \frac{\rho}{P}(\rho P')' + n^2\rho^2 = -\frac{\Phi''}{\Phi} = f(\varphi)$$

So $\quad \frac{d^2\Phi''}{d\varphi^2} = -m^2\Phi$

or $\quad \Phi_m(\varphi) = C_m \sin m\varphi + D_m \cos m\varphi.$

And

$$\rho\,(\rho P')' + (n^2\rho^2 - m^2)\,P = 0$$

which is Bessel's equation. (Avoid at all cost.) The solutions are

$$A_m J_m(n\rho) + B_m J_{-m}(n\rho),$$

and so

$$\psi(\rho,\phi,z) = \sum_{\substack{n,m,\ell \\ n^2\ell^2 = k^2}} \left[A_m J_m(n\rho) + B_m J_{-m}(n\rho)\right]$$

$$\times\left[C_m \sin m\phi + D_m \cos m\phi\right]$$

$$\times\left[E_\ell e^{\ell z} + F_\ell e^{-\ell z}\right].$$

m is an integer if ψ is to be single valued

$\downarrow$

(r,θ,ϕ): B(2.461), we have.

$$\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\right.$$
$$\left.+\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]=-h^2\psi$$

$$\psi(r,\theta,\phi)=R(r)\,\Theta(\theta)\,\Phi(\phi)$$

$$\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$
$$+\frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2}=-h^2 \quad \text{or}$$

$$h(\phi)=\frac{1}{\Phi}\Phi''=r^2\sin^2\theta\left[-h^2-\frac{1}{r^2R}(r^2R')'-\frac{1}{r^2\sin\theta\Theta}(\sin\theta\Theta')'\right]$$

$$\frac{\Phi''}{\Phi}=-m^2 \qquad \phi''=-m^2\phi^2$$

$$\phi_m(\phi)=E_m\sin m\phi+F_m\cos m\phi$$

m an integer if ψ is single valued.

ψ must be single valued if $0\le\phi\le2\pi$ is physical.

$$\frac{1}{r^2 R}(r^2 R')' + \frac{1}{r^2 \sin\theta\, \Theta}(\sin\theta\, \Theta')' - \frac{m^2}{r^2 \sin^2\theta} = -k^2$$

$$+Q = \frac{1}{R}(r^2 R')' + r^2 k^2 = -\frac{1}{\sin\theta\, \Theta}(\sin\theta\, \Theta')' + \frac{m^2}{\sin^2\theta}$$

$$\frac{1}{\sin\theta}(\sin\theta\, \Theta')' - \frac{m^2}{\sin^2\theta}\Theta + Q\Theta = 0$$

$$\frac{1}{r^2}(r^2 R')' + k^2 R - \frac{QR}{r^2} = 0$$

Need $Q = \ell(\ell+1)$, $\ell \geq$ an integer, ~~and then~~ if $0 \leq \theta \leq \pi$ is physical

$$Q(\theta) = P_\ell^m(\cos\theta) \qquad (12.81)$$

and if $k^2 > 0$, then

$$R(r) = j_\ell(kr) \quad \text{if } 0 \leq r \leq \infty$$

is physical. If ~~for~~ $r=0$ is not part of the physical domain, then

$$R(r) = n_\ell(kr) \text{ also works.}$$

These spherical Bessel functions are nice:

$$j_0(x) = \frac{\sin x}{x} \qquad n_0(x) = -\frac{\cos x}{x}.$$

Some sp[redacted]al c[redacted]:

If $k^2 = 0$, then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \frac{\ell(\ell+1)R}{r^2}$$

is just

$$(r^2 R')' = \ell(\ell+1)R$$

and if $R = r^\alpha$, then

$$(\alpha r^2 r^{\alpha-1})' = (\alpha r^{\alpha+1})' = \alpha(\alpha+1)r^\alpha$$

$$= \ell(\ell+1) r^\alpha \text{ is solved by}$$

$$\alpha(\alpha+1) = \ell(\ell+1)$$

$$\alpha = \ell \qquad \text{so}$$

$$\alpha^2 + \alpha - \ell(\ell+1) = (\alpha-\ell)(\alpha+\ell+1)$$

$$\alpha = -\ell-1 \qquad \text{So}$$

$$R(r) = A_\ell r^\ell + B_\ell r^{-\ell-1}.$$

Thus the spherically symmetric solution of Laplace's eq.

$$\Delta \phi = 0$$

that is regular at $r = \infty$, has $\ell = 0$ and is

$$\phi(r) = B_0 r^{-1}$$

which is the field of a point charge B_0.

If the problem has azimuthal symmetry, $m = 0$, and then

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos\theta).$$

If there's no symmetry at all and $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ are physical, then

$$\phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell^m(\cos\theta)$$

$$\left(D_m \cos m\phi + E_m \sin m\phi \right).$$

or

$$\phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) Y_\ell^m(\theta, \phi)$$

$Y_n^m(\theta, \phi)$ is defined by (12.144–147).

$$\delta(\Omega-\Omega') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^{m*}(\Omega')Y_l^m(\Omega)$$

$$\int d\Omega\, Y_{l'}^{m'*}(\theta,\phi)\, Y_l^m(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

These are complete on unit sphere.

$d\Omega = \sin\theta\, d\theta\, d\phi$. For example,

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\hat{r}\cdot\hat{r}') \quad \text{or}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r_>}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{r_<}{r_>}\right)^l \frac{4\pi}{2l+1} Y_l^{m*}(\theta',\phi')Y_l^m(\theta,\phi).$$

where $r_> = \max(|\vec{r}|, |\vec{r}'|)$

$r_< = \min(|\vec{r}|, |\vec{r}'|)$.

The equation

$-\Delta\psi = k^2\psi$ is separable even if

$$k^2 = f(r) + \frac{g(\theta)}{r^2} + \frac{h(\phi)}{r^2\sin^2\theta} + k'^2.$$

The hydrogen atom:

$$\left[-\frac{\hbar^2}{2m}\Delta - \frac{Ze^2}{r}\right]\psi = E\psi$$

is an example.

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2\, e^{-Zr/a_0}$$

$$\langle r \rangle = \left(\frac{a_0}{2Z}\right)\left[3n^2 - \ell(\ell+1)\right]$$

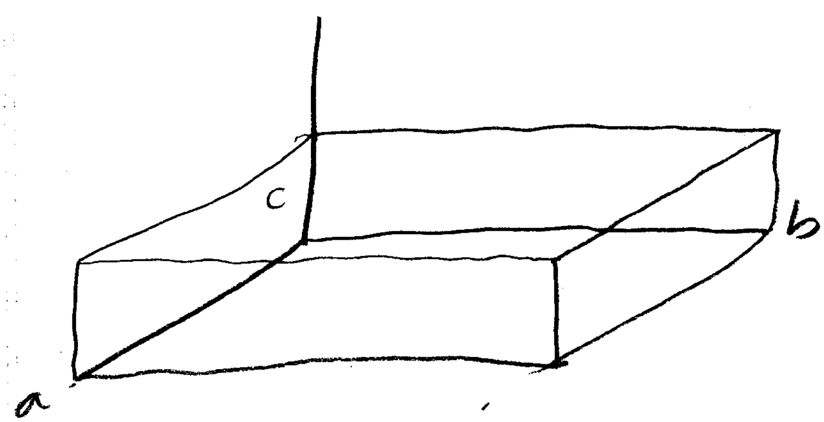
So $\langle r \rangle_{10} = \frac{a_0}{2Z} 3 = \frac{3}{2Z} a_0$.

When x, y, z work, they are simple.

Prob. 8.3.5 Particle in a box

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi$$

↑ sides a, b, c



$$E = E_a + E_b + E_c \qquad \psi = XYZ$$

$$-\frac{\hbar^2}{2m}\frac{d^2X}{dx^2} = E_a X$$

$$X(0) = X(a) = 0$$

$$X(x) = \sin \pi \frac{x}{a} n$$

$$-\frac{\hbar^2}{2m}\left(-\frac{\pi n}{a}\right)^2 \sin\frac{\pi x n}{a} = E_a \sin\frac{\pi n x}{a}$$

$$E_a = \frac{\hbar^2}{2m}\left(\frac{\pi n}{a}\right)^2$$

$$E_b = \frac{\hbar^2}{2m}\left(\frac{\pi m}{b}\right)^2 ; \quad E_c = \frac{\hbar^2}{2m}\left(\frac{\pi l}{c}\right)^2$$

$$E_{nml} = \frac{(\pi\hbar)^2}{2m}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}\right)$$

ground state is

$$E_{000} = \frac{h^2}{8m}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

Application of Green's function.

$$\Phi(\vec{x}) = \int \frac{\rho(\vec{x'})\, d^3x'}{|x-x'|}$$

satisfies $\nabla\cdot(-\nabla\Phi) = 4\pi\rho$.

$$\frac{1}{|x-x'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1} Y^*_{lm}(\Omega')Y_{lm}(\Omega)\frac{r_<^l}{r_>^{l+1}}$$

So

$$\Phi(\vec{x}) = 4\pi\sum_{lm}\frac{1}{2l+1}\left[\int Y_{l}^{m*}(\Omega') r'^l \rho(x')\, d^3x'\right] \times \frac{Y_{lm}(\Omega)}{r^{l+1}}$$

if the point $\vec{x}$ is outside the charged region $|\vec{x}| > |\vec{x}'|$ when $\rho(x') \neq 0$.

The last homework ~~problem~~:

$$\frac{p^2}{2m}\psi = E\psi \qquad \psi(a,\theta,\phi) = 0$$

$$\vec{p} \to \frac{\hbar}{i}\vec{\nabla} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$-\frac{\hbar^2}{2m}\Delta\psi = E\psi \qquad -\Delta\psi = \frac{2mE}{\hbar^2}\psi = k^2\psi$$

(2.46) $$k^2\psi = -\Delta\psi = -\frac{1}{r^2}(r^2R')'Y - \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} =$$

$$N\frac{e^{im\phi}}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\ell^m}{\partial\theta}\right)\right) - \frac{m^2N}{\sin^2\theta}P_\ell^m e^{im\phi}$$

(12.146) since $Y_\ell^m = N P_\ell^m(\cos\theta)\, e^{im\phi}$. Now P_ℓ^m satisfies

(12.71) $$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP_\ell^m(\cos\theta)}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right]P_\ell^m(\cos\theta) = 0$$

So

$$k^2\psi = -\frac{1}{r^2}(r^2R')'Y_{\ell m} + \frac{\ell(\ell+1)}{r^2}RY_{\ell m} = k^2 RY_{\ell m}$$

So

$$-(r^2R')' + [\ell(\ell+1) - h^2r^2]R = 0.$$

A quicker route is to use Ex. 2.5.16

$$-\Delta = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\vec{L}^2}{r^2\hbar^2}$$ where

$$L = r \times p = \frac{\hbar}{i} r \times \nabla.$$

The Y's satisfy

$$L^2 Y_\ell^m = \hbar^2 \ell(\ell+1) Y_\ell^m.$$

So

$$-\Delta R Y_\ell^m = -\frac{1}{r^2}\partial_r(r^2\partial_r R)Y_\ell^m + \frac{R}{\hbar^2}\frac{L^2 Y_\ell^m}{r^2}$$

$$= -\frac{1}{r^2}(r^2R')'Y_\ell^m + R\frac{\ell(\ell+1)}{r^2}Y_\ell^m = h^2 R Y_\ell^m$$

$$-(r^2R')' + [\ell(\ell+1) - r^2h^2]R = 0$$

(11.139) $$(r^2R')' + [r^2h^2 - \ell(\ell+1)]R = 0$$

the solutions regular at $r=0$ are

$$R_\ell(r) = j_\ell(kr).$$

Now we want $R_\ell(a) = 0$,

$$0 = j_\ell(ka) \quad \text{so} \quad ka = \alpha_{\ell n}$$

where $j_\ell(\alpha_{\ell n}) = 0$. The $\alpha_{\ell n}$ are the roots of j_ℓ. So

$$k_{\ell n} = \frac{\alpha_{\ell n}}{a} \quad \text{and so}$$

$$E_{\ell n} = \frac{(\hbar k)^2}{2m} = \frac{(\hbar \alpha_{\ell n})^2}{2m a^2}.$$

The function $j_0(x) = \frac{\sin x}{x}$ has its roots at $\alpha_{0n} = n\pi \qquad n = 1, 2, \cdots$.

So the s-state levels are

$$E_{0n} = \frac{(\hbar n\pi/a)^2}{2m}.$$

Say $m = m_e$ and $a = 1\,\text{Å}$. Then

$$E_{0n} = \frac{n^2\pi^2}{2mc^2}\,\frac{\hbar^2 c^2}{a^2} \simeq \frac{n^2\pi^2}{2mc^2}\,\frac{(200\,\text{MeV f})^2}{10^{+10}\,\text{f}^2} = \frac{n^2\pi^2}{2mc^2}(2000\,\text{eV})^2$$

$$= \frac{n^2\pi^2\, 4\,\text{MeV eV}}{1\,\text{MeV}} = 4n^2\pi^2\,\text{eV}.$$

The H atom

$$\left[\frac{p^2}{2m} - \frac{Ze^2}{r}\right]\psi = E\psi$$

$$\left[-\frac{\hbar^2}{2m}\Delta - \frac{Ze^2}{r}\right]\psi = E\psi$$

$$\left[-\Delta - \frac{2mZe^2}{\hbar^2}\frac{1}{r}\right]\psi = E\psi$$

$$\left[-\frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{L^2}{\hbar^2 r^2} - \frac{2mZe^2}{\hbar^2}\frac{1}{r}\right]RY = ERY_\ell^m$$

(13.54) $$-Y\frac{1}{r^2}(r^2R')' + \frac{\ell(\ell+1)}{r^2}RY_\ell^m - \frac{2mZe^2}{\hbar^2}\frac{RY}{r} = ERY$$

$$(r^2R')' - \ell(\ell+1)R + \frac{2mZe^2}{\hbar^2}rR = -r^2ER$$

$$\rho = \alpha r \qquad \alpha^2 = \frac{8m|E|}{\hbar^2} \qquad \lambda = \frac{2mZe^2}{\alpha\hbar^2}$$

$$\frac{1}{\rho^2}(\rho^2\chi')' + \left(\frac{\lambda}{\rho} - \frac{1}{4} - \frac{\ell(\ell+1)}{\rho^2}\right)\chi = 0$$

$$\chi(\rho) = R(\rho/\alpha).$$

let

$$\rho = \left(\frac{8m|E|}{\hbar^2}\right)^{\frac{1}{2}} r .$$

Then

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi) \quad \text{where}$$

$$R_{nl}(r) = -\left\{ \left(\frac{2Z}{na_0}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3} \right\}^{1/2}$$

$$e^{-\rho/2} \rho^l L_{n+l}^{2l+1}(\rho)$$

$$E_n = -\frac{Z^2e^2}{2n^2a_0} = -\frac{1}{2} mc^2 \frac{\alpha^2}{n^2} Z^2$$

where $\alpha = e^2/\hbar c$ and $a_0 = \frac{\hbar^2}{me^2} \sim 0.529 \mathring{A}$

$n \geqslant l+1$ $\qquad \rho = \frac{2Zr}{na_0}$

$$L_n^q(\rho) = \frac{d^q}{d\rho^q} L_n(\rho) \qquad \text{Laguerre}$$

$$L_n(\rho) = e^\rho \frac{d^n}{d\rho^n}\left(\rho^n e^{-\rho}\right)$$

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2 e^{-Zr/a_0}$$

$$\langle r \rangle = \frac{a_0}{2Z}\left[3n^2 - l(l+1)\right]$$

$$\langle r \rangle_{10} = \frac{a_0}{2Z} 3 = \frac{3}{2Z} a_0$$

$$a_0 \cong 0.529 \text{ Å}$$

Linear Algebra

Suppose we consider the system

$$a_{11} x_1 + a_{12} x_2 = 0$$

$$a_{21} x_1 + a_{22} x_2 = 0 ,$$

Then the vector $x = (x_1, x_2)$ must be $\perp$ to both vectors
$a_1 = (a_{11}, a_{12})$ and $a_2 = (a_{21}, a_{22})$:

$$\vec{a}_1 \cdot \vec{x} = \vec{a}_2 \cdot \vec{x} = 0 .$$

This works only if a_1 and a_2 are $\parallel$:

$$a_1 = \lambda a_2 \quad \text{or} \quad (a_{11}, a_{12}) = \lambda (a_{21}, a_{22}).$$

so that $a_{11} a_{22} = \lambda a_{21} \dfrac{a_{12}}{\lambda} = a_{21} a_{12}$ i.e.

$$a_{11} a_{22} - a_{21} a_{12} = 0.$$

In matrix notation, the system of equations is

$Ax = 0$. It has a solution only if the determinant of the matrix A

$$|A| = a_{11} a_{22} - a_{21} a_{12} = 0$$

vanishes.

Similarly the 3 equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

or $\quad A x = 0 \quad$ with $\quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

has a solution only if $\vec{x}$ is $\perp$ to all three vectors

$$a_i = (a_{i1}, a_{i2}, a_{i3}) \qquad \text{for } i = 1, 2, 3.$$

But any vector $\perp$ to $\vec{a}_1$ and $\vec{a}_2$ must be a multiple of $\vec{a}_1 \times \vec{a}_2$. So

$$\vec{x} = \lambda\, \vec{a}_1 \times \vec{a}_2 .$$

Then

$$0 = \vec{a}_3 \cdot \vec{x} = \lambda (a_1 \times a_2) \cdot a_3$$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}$$

$$= \det A .$$

We will see that in general for $n \times n$ matrices A, the system

$0 = A x$ or $0 = \sum_{j=1}^{n} a_{ij} x_j$ has a solution only if