Review

If

$f(x,y) = f(z)$, $z = x+iy$

is differentiable, then

$$f(x,y) = u(x,y) + i\,v(x,y)$$

u, v real.

$$df = du + i\,dv = f'(z)\,dz = f'(z)(dx + i\,dy)$$

$$= (u' + i v')(dx + i\,dy).$$

Then

$$u_x = u' = v_y \qquad \text{and}$$

$$-u_y = v' = v_x .$$

These are the C-R conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

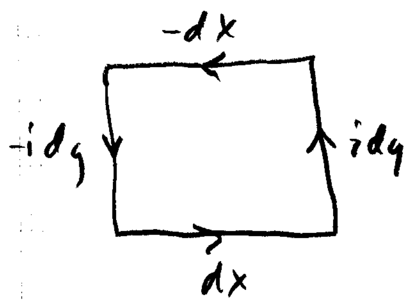
$f(z)$ is <u>analytic</u> at $z = z_0$ if it is differentiable at $z = z_0$ and in a small region around z_0.

If $f(z)$ is analytic for all z, then it is <u>entire</u>.

Contour integrals.

$$\int f(z)\, dz = \lim_{N \to \infty} \sum_{i=1}^{N-1} f(z_i)\,(z_{i+1} - z_i)$$

Say $f(z)$ is analytic in a region and so satisfies the C-R conditions there.



$$\oint f(z)dz = (u+iv)dx - (u+iv)idy$$

$$+ (u + u_x dx + iv + iv_x dx)\, idy$$

$$- (u + u_y dy + iv + iv_y dy)\, dx \qquad \text{so}$$

$$\oint f(z)dz = (iu_x - v_x - u_y - iv_y)\, dx\,dy = 0 \quad \text{if}$$

$$u_x = v_y \quad \text{and} \quad -v_x = u_y.$$

In general

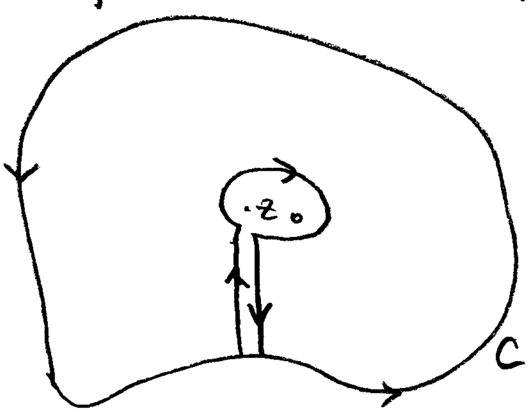
$$\oint f(z)\,dz = 0$$

as long as the contour lies within a region in which $f(z)$ is analytic and in which one could shrink the contour to a point.

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\,dz$$

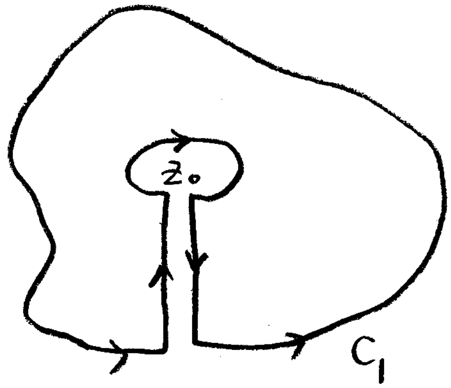
where z_0 and the contour are within a (simply connected) region of analyticity.



$$\frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)\,dz}{z - z_0} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)\,dz}{z - z_0}$$

$$C = C_1 - C_{\uparrow\downarrow} - C_{Q} = C_{\circlearrowright}$$

Now the integral along C_1 is



which is a closed curve in a region with no holes in which $f(z)/(z-z_0)$ analytic.

So
$$\oint_{C_1} \frac{f(z)}{z-z_0} = 0.$$

The integral along C_2 just cancels the clockwise integral around z_0. So

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)\,dz}{z-z_0}$$

in which C is the more-or-less arbitrary counter-clockwise contour around z_0 and C_2 is a tight c-cw circle around z_0. So

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi i}\int_0^{2\pi} d\theta\, \frac{[f(z_0) + f'(z_0)\, re^{i\theta}]\, i re^{i\theta}}{re^{i\theta}}$$

where $z - z_0 = re^{i\theta}$ $\quad z = z_0 + re^{i\theta}$

$dz = i re^{i\theta} d\theta$.

So

$$\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[f(z_0) + f'(z_0)\, r e^{i\theta} \right]$$

$$= f(z_0).$$

This is Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

in which C is a c-c-w contour around z_0 that lies within a hole-free region of analyticity for $f(z)$.

$$\frac{f(z_0+dz_0) - f(z_0)}{dz_0} = \frac{1}{2\pi i\, dz_0}\oint f(z)\,dz \left(\frac{1}{z-z_0-dz_0} - \frac{1}{z-z_0} \right)$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0-dz_0)(z-z_0)}$$

So taking the limit $dz_0 \to 0$, we get

$$f'(z_0) = \frac{1}{(2\pi i)}\oint \frac{f(z)\,dz}{(z-z_0)^2}.$$ Next

$$f^{(2)}(z_0) = \frac{2}{(2\pi i)}\oint \frac{f(z)\,dz}{(z-z_0)^3}$$

In general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

So if $f(z)$ is analytic in a region, it is also infinitely differentiable there.

If $f(x,y)$ is continuous in a region and every closed integral there vanishes,

$$0 = \oint f(x,y)\,dz,$$

then $f(z) = f(x,y)$ is analytic there.

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$

is independent of the contour and also

$$\frac{dF(z)}{dz} = f(z).$$

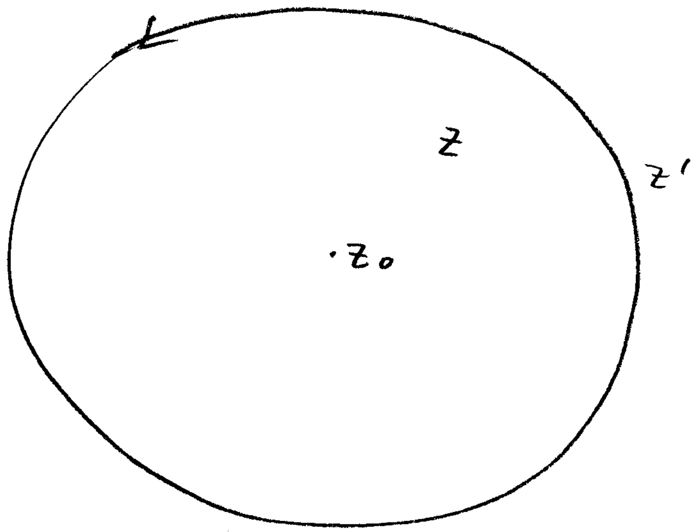
So $F(z)$ is analytic there, Thus $f(z)$ is too.

Taylor expansion

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z_0 \;-(z-z_0)}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z_0\left[1 - \frac{z-z_0}{z'-z_0}\right]}$$



Now $\quad \left|\frac{z-z_0}{z'-z_0}\right| < 1 \quad$ so

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n$$

converges absolutely and uniformly.

So we may integrate the series

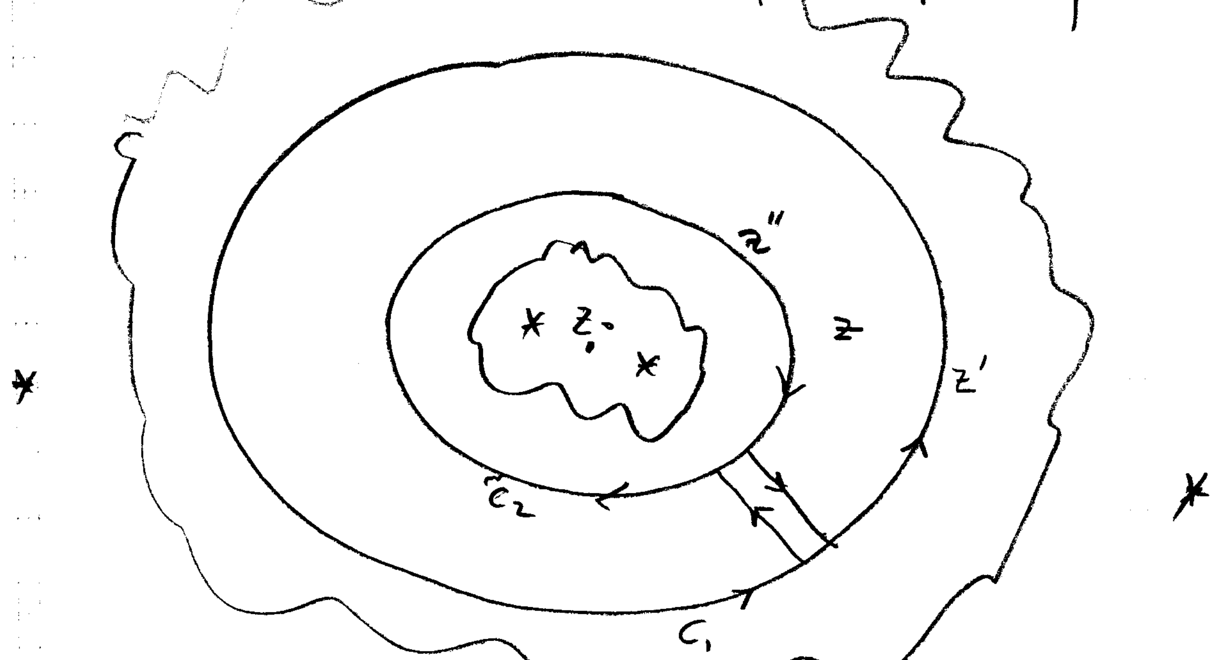
$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')dz'}{z'-z_0} \sum \left(\frac{z-z_0}{z'-z_0}\right)^n$$

term by term

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint \frac{f(z')dz'}{(z'-z_0)^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

Now consider an annulus of analyticity



and a closed contour C which is equivalent to a c-c-w contour C_1 and a c-w contour $\tilde{C}_2$. Let $C_2 = -\tilde{C}_2$ be the c-c-w version of $\tilde{C}_2$.

Then

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')\,dz''}{z''-z}$$

Now if z_0 is the center of both circular contours C_1 and C_2, then

$$\left| \frac{z''-z_0}{z-z_0} \right| < 1 \text{ on } C_2 \;\&\; \left| \frac{z-z_0}{z'-z_0} \right| < 1 \quad \text{on } C_1$$

So

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\,dz'}{z'-z_0-(z-z_0)} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')\,dz''}{z-z_0-(z''-z_0)}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\,dz'}{(z'-z_0)\left[1 - \frac{z-z_0}{z'-z_0}\right]} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z'')\,dz''}{(z-z_0)\left[1 - \frac{z''-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i} \oint_{C_1} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}} f(z')\,dz' + \frac{1}{2\pi i} \oint_{C_2} \sum_{n=0}^{\infty} \frac{(z''-z_0)^n}{(z-z_0)^{n+1}} f(z'')\,dz''$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_{C_1} \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(z-z_0)^n} \frac{1}{2\pi i} \oint_{C_2} f(z'')\,dz''\,(z''-z_0)^{n-1}$$

Now the functions

$$\frac{f(z')}{(z'-z_0)^{n+1}} \qquad \text{and} \qquad f(z'')(z''-z_0)^{n-1}$$

are analytic in the annulus, so we may displace the contours to a common contour C:

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$+ \sum_{m=-\infty}^{-1} (z-z_0)^m \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{m+1}} \, .$$

Let

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \, ,$$

then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \text{ is}$$

the Laurent series for $f(z)$ in the annulus.

The $*$'s represent possible singularities. If they are merely poles, then $f(z)$ is meromorphic. A pole of order n is

$$\frac{1}{(z-z_*)^n} \, .$$

Example

$$f(z) = \frac{1}{z(z+1)}$$

set $z_0 = 0$, $r = 0$, $R = 1$,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{dz}{z(z+1)z^{n+1}}$$

where C is circle $0 < |z| < 1$.

$$a_n = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+2}(z+1)}$$

$$= \frac{1}{2\pi i} \oint \sum_{m=0}^{\infty} (-z)^m \frac{dz}{z^{n+2}}$$

$$z = \rho e^{i\theta} \qquad dz = i\rho e^{i\theta} d\theta$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} d\theta \; \frac{\sum (-1)^m \rho^m e^{im\theta}}{\rho^{n+2} e^{i(n+2)\theta}} \, i\rho e^{i\theta} d\theta$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} d\theta \sum (-1)^m \rho^{m-n-1} e^{i(m+n-1)\theta} \quad d\theta$$

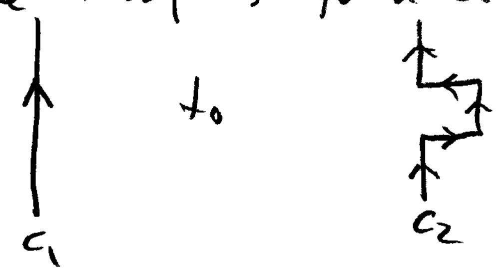
$$= \sum_{m=0}^{\infty} \rho^{m-n-1} (-1)^m \delta_{m-n-1,0} = (-1)^{n+1}$$

Note that much of c.v. theory follows from the vanishing of the integral of an analytic function around a tiny square.

$$0 = \oint_{\square} f(z)\, dz$$

Thus we may shift a contour

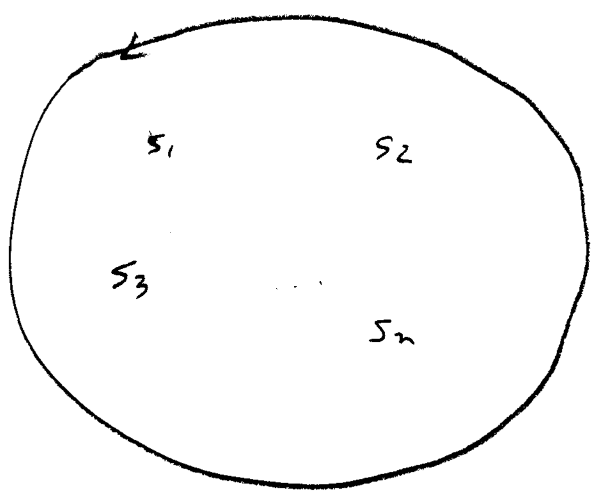
from



because $C_2 - C_1 = \square$ and $\oint_{\square} f(z)\, dz = 0$.

By iterating such shifts by one pixel at a time, one may move a contour quite generally throughout a region of analyticity without changing the value of the integral.

Now suppose we have a contour that encloses n isolated singularities in a region in which $f(z)$ is analytic.



Then by shrinking the contour, we get

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{n} 2\pi i\, a_{-1}(z_k)$$

in which we've used the moronic notation

$$a_{-1}(z_k) = \frac{1}{2\pi i} \oint_{C_k} f(z)\,dz .$$

These integrals $a_{-1}(z_k)$ are called residues.

So

$$\oint_C f(z)\,dz = 2\pi i \sum a_{-1}(z_k)$$

$$= 2\pi i \text{ (sum of enclosed residues)},$$

This is the residue theorem.

Examples of the calculus of residues:

Consider the integral

$$I(t,m) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k^2+m^2} .$$

$I(t,m)$ is a Fourier transform of $f(k) = (k^2+m^2)^{-1}$.

First assume $t>0$. Then we may close the contour by adding a hemispheric contour

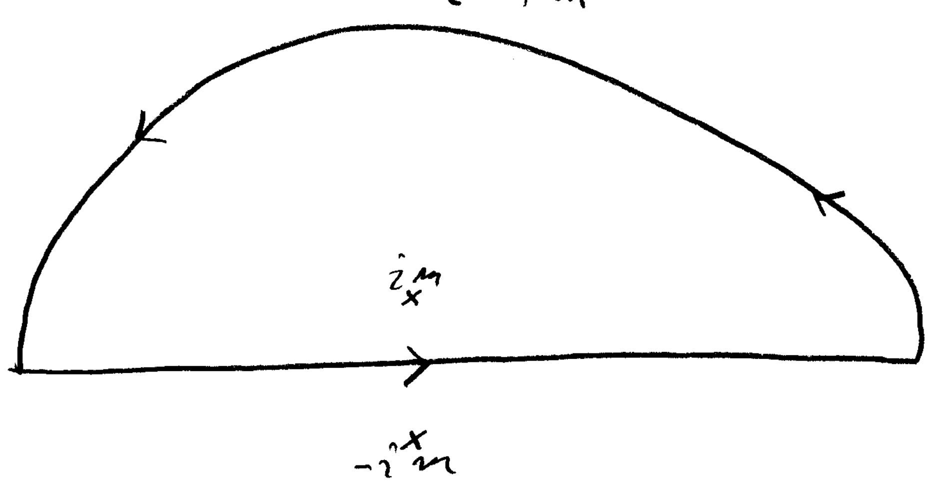
$$z = R\, e^{i\theta} \qquad 0 \leq \theta \leq \pi$$

and letting $R \to \infty$. Now for $k = z = Re^{i\theta}$,

$$e^{ikt} = e^{iRt(\cos\theta + i\sin\theta)} = e^{iRt\cos\theta} \, e^{-Rt\sin\theta} .$$

So for $t>0$, $e^{ikt} \to 0$ as $R \to \infty$. So by adding the extra hemispheric contour, we do not change $I(t,m)$:

$$I(t,m) = \oint dz \, \frac{e^{izt}}{z^2+m^2}$$



There is one pole within the contour because

$$\frac{e^{izt}}{z^2+m^2} = \frac{e^{izt}}{(z-im)(z+im)} .$$

So we may shrink the contour to a tiny circle around $z = im$:

$$I(t,m) = \oint dz \, \frac{e^{izt}}{(z+im)(z-im)}$$

see 241.1

$$= 2\pi i \left. \frac{e^{izt}}{z+im} \right|_{z=im}$$

$$= \frac{2\pi i}{2im} e^{i\,imt} = \frac{\pi}{m} e^{-mt} .$$

What if $t < 0$? Then we may close the contour by adding a lower hemispheric contour

$$k = z = Re^{i\theta} \qquad \pi \le \theta \le 2\pi .$$

The integral

$$\int dz \, \frac{e^{izt}}{z^2+m^2}$$

Here we use the basic formula

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z - z_0}$$

for the case

$$f(z) = 2\pi i \frac{e^{izt}}{z + im}$$

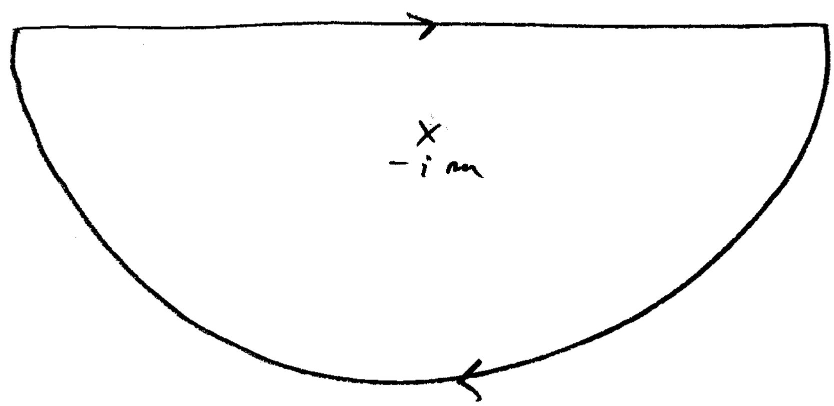
and $z_0 = im$.

is zero because for $t<0$

$$e^{izt} = e^{iRt(\cos\theta + i\sin\theta)} = e^{iRt\cos\theta}\, e^{-R|t|\sin\theta|}$$

vanishes as $R \to \infty$. So

$$I(t,m) = \oint \frac{e^{izt}}{(z-im)(z-(-im))}$$



is a huge closed contour, which enclosed one pole. But this contour is clockwise, so

$$I(t,m) = -2\pi i \left(\frac{e^{izt}}{z-im} \right) \Bigg|_{z=-im}$$

$$= \frac{-2\pi i}{-2im} e^{+mt} = \frac{\pi}{m} e^{+mt} \quad \text{for } t<0.$$

Combining the two results, for $t>0$ and $t<0$, we get

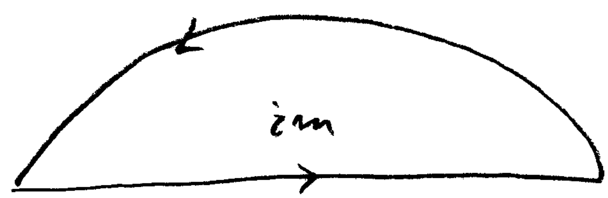
$$\frac{\pi}{m} e^{-m|t|} = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k^2+m^2}.$$

As a second example, let's consider the integral

$$J(t,m) = \int_{-\infty}^{\infty} dk\, \frac{e^{ikt}}{(k^2+m^2)^2} = \int_{-\infty}^{\infty} dk\, \frac{e^{ikt}}{(k-im)^2(k-(-im))^2},$$

which is the Fourier transform of $(k^2+m^2)^{-2}$.
So for $t>0$, we add the "northern" contour

$$J(t,m) = \oint dz\, \frac{e^{izt}}{(z-im)^2(z-(-im))^2}$$



$$J(t,m) = \oint dz\, \frac{e^{izt}}{(z+im)^2(z-im)^2}$$

See p. 243.1

$$= 2\pi i \left[\frac{d}{dz} \frac{e^{izt}}{(z+im)^2} \right]\Bigg|_{z=im}$$

See p. 243.1 or Eq. (6.47) for $m=1$. So for $t>0$,

$$J(t,m) = 2\pi i \left[\frac{i t e^{-mt}}{(2im)^2} - \frac{2 e^{-mt}}{(2im)^3} \right]$$

$$= 2\pi i\, e^{-mt} \left(-\frac{it}{4m^2} - \frac{2i}{8m^3} \right)$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

for $n=1$

$$f^{(1)}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^2}$$

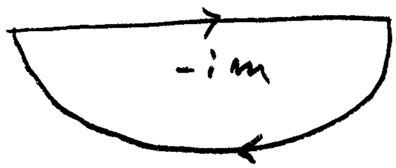
We have $z_0 = im$

$$f(z) = 2\pi i \, \frac{e^{izt}}{(z+im)^2}$$

$$J(t,m) = \frac{2\pi e^{-mt}}{4m^2}\left(t + \frac{1}{m}\right)$$

$$= \frac{\pi}{2m^2}\left(t+\frac{1}{m}\right)e^{-mt} \qquad \text{for } t>0.$$

For $t<0$, we add a "southern" contour



So for $t<0$,

$$J(t,m) = \oint dz \frac{e^{izt}}{(z-im)^2(z-(-im))^2}$$

$$= -2\pi i \left[\frac{d}{dz}\frac{e^{izt}}{(z-im)^2}\right]\Bigg|_{z=-im}$$

$$= -2\pi i\left[\frac{it\,e^{mt}}{(-2im)^2} - \frac{2\,e^{mt}}{(-2im)^3}\right]$$

$$= -\frac{\pi i e^{mt}}{2m^2}\left(-it + \frac{i}{m}\right) = \frac{\pi}{2m^2}\left(\frac{1}{m} - t\right)e^{mt}$$

for $t<0$.

Combining the two cases, $t>0$ & $t<0$, we get

$$\int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{(k^2+m^2)^2} = \frac{\pi}{2m^2}\left(|t| + \frac{1}{m}\right) e^{-m|t|}$$

where $m>0$.

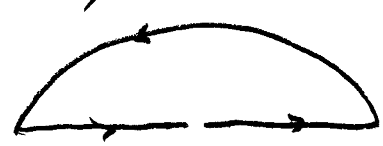
What if there is a simple pole on the real axis? For instance, consider

$$L(t,m) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)}$$

which is the Fourier transform of $[k(k-im)]^{-1}$.
$L(t,m)$ is the limit as $\epsilon \to 0$:

$$L(t,m) = \int_{-\infty}^{-\epsilon} dk \, \frac{e^{ikt}}{k(k-im)} + \int_{\epsilon}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)}.$$

Now we'll want to make this a closed contour.
So for $t>0$, we add the "northern" loop,



but we still have the gap at $k=0$. We may add and subtract either tiny contour,

 or .

Let's add and subtract $\frown$:

$$L(t,m) = \oint dz \, \frac{e^{izt}}{z(z-im)} - \int_{\frown} dz \, \frac{e^{izt}}{z(z-im)}$$

The pole $\frac{1}{z}$ at $z=0$ is now outside the contour. The tiny contour is

$$\int_{\frown} dz \, \frac{e^{izt}}{z(z-im)} = \frac{1}{-im} \int_{\frown} \frac{dz}{z}$$

Let $z = \epsilon e^{i\theta}$ $\quad dz = i \epsilon e^{i\theta} d\theta$

$$\int_{\frown} dz \, \frac{e^{izt}}{z(z-im)} = \frac{i}{m} \int_{\pi}^{0} d\theta \, \frac{i \epsilon e^{i\theta}}{\epsilon e^{i\theta}} = -\frac{1}{m} \int_{\pi}^{0} d\theta = \frac{\pi}{m}$$

So

$$L(t,m) = \oint dz \, \frac{e^{izt}}{z(z-im)} - \frac{\pi}{m}$$

$$= 2\pi i \, \frac{e^{-mt}}{im} - \frac{\pi}{m}$$

$$= \frac{2\pi}{m} e^{-mt} - \frac{\pi}{m} = \frac{\pi}{m}\left(2 e^{-mt} - 1\right).$$

Suppose we had included the residue at $z = 0$. That would have given

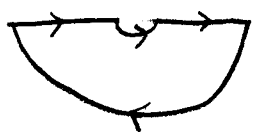
$$2\pi i \frac{1}{-im} = -\frac{2\pi}{m}.$$

So we see that the pole on the real axis counts with weight $1/2$.

The addition and subtraction of $\curvearrowright$ gives the same result.

Now we do it for $t<0$:

$$L(t,m) = \oint dz \frac{e^{izt}}{z(z-im)} - \int_{\curvearrowright} dz \frac{e^{izt}}{z(z-im)}$$



this integral vanishes, and so

$$L(t,m) = -\frac{1}{(-im)} \int_{\pi}^{2\pi} d\theta \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}$$

$$= -\frac{i}{m}\, i \int_{\pi}^{2\pi} d\theta = \frac{\pi}{m}$$ which is $1/2$

the residue

$$-2\pi i \frac{1}{-im} = +\frac{2\pi}{m}$$

had we included the pole $\frac{1}{z}$ at $z=0$

by going over it in a clockwise sense

Combining the two results, we have

$$\int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)} = \begin{cases} \frac{\pi}{m}\left(2e^{-mt} - 1\right) & t > 0 \\ \frac{\pi}{m} & t < 0 \,. \end{cases}$$

A simple rule for handling simple poles on contours is to go around them both ways and then to average the two answers.

Consider the $\epsilon \to 0$ limit of the integral

$$I = \int_{-\infty}^{\infty} \frac{f(x)}{x - i\epsilon}\, dx,$$

where $f(x)$ is smooth, or the integral

$$\mathcal{I} = \int_{-\infty}^{\infty} \frac{f(z)}{z - i\epsilon}\, dz ,$$

where $f(z)$ is analytic.

$$\mathcal{I} = \int_{-\infty}^{-\delta} \frac{f(z)\,dz}{z - i\epsilon} + \int_{\delta}^{\infty} \frac{f(z)\,dz}{z - i\epsilon} + \int_{-\delta}^{\delta} \frac{f(z)\,dz}{z - i\epsilon}$$

Now we let $\epsilon \to 0$ in the first two integrals:

$$\mathcal{I} = \int_{-\infty}^{-\delta} \frac{f(z)\,dz}{z} + \int_{\delta}^{\infty} \frac{f(z)\,dz}{z} + \int_{-\delta}^{\delta} \frac{f(z)\,dz}{z - i\epsilon}$$

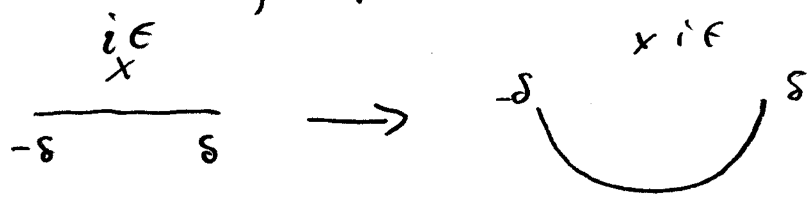
The <u>Cauchy</u> <u>principal</u> <u>value</u> is defined as

$$\lim_{\delta \to 0} \int_{-\infty}^{-\delta} \frac{f(z)\,dz}{z} + \int_{\delta}^{\infty} \frac{f(z)\,dz}{z} = P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z}.$$

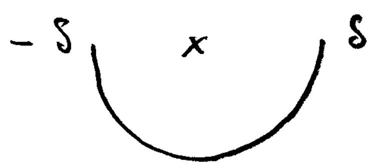
So

$$\mathcal{I} = P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z} + \lim_{\substack{\delta \to 0 \\ \epsilon \to 0}} \int_{-\delta}^{\delta} \frac{f(z)\,dz}{z - i\epsilon}$$

Now we may deform the contour



and now we may take the $\epsilon \to 0$ limit:



So now

$$\mathcal{I} = P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z} + \int dz\,\frac{f(z)}{z}$$

f is continuous

$$= P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z} + f(0)\int \frac{dz}{z}$$

$$z = \delta e^{i\theta} \qquad \theta : [\pi, 2\pi] \qquad dz = i\delta e^{i\theta}\,d\theta$$

$$\mathcal{I} = P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z} + f(0)\int_{\pi}^{2\pi} \frac{d\theta\, i\delta e^{i\theta}}{\delta e^{i\theta}}$$

$$= P\int_{-\infty}^{\infty} \frac{f(z)\,dz}{z} + i f(0)\int_{\pi}^{2\pi} d\theta$$

$$= P\int_{-\infty}^{\infty} \frac{f(z)}{z}\,dz + i\pi f(0).$$

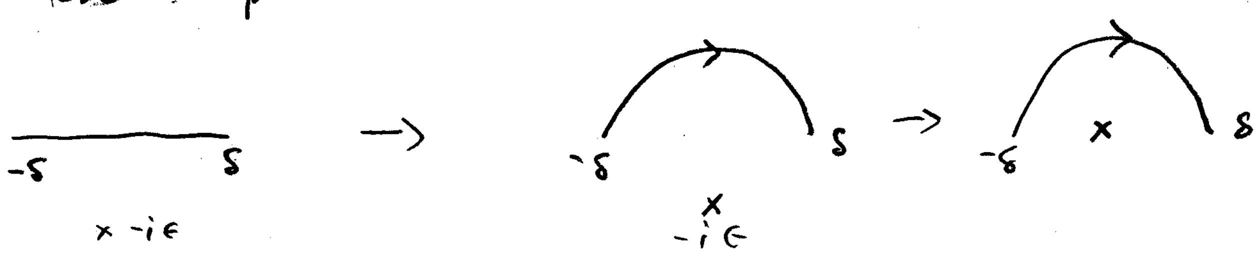
Physicists write this result as

$$\frac{1}{z - i\epsilon} = P\,\frac{1}{z} + i\pi\,\delta(z).$$

Letting $z = x - x_0$, one has

$$\frac{1}{x - x_0 - i\epsilon} = P\,\frac{1}{x - x_0} + i\pi\,\delta(x - x_0).$$

The case $\frac{1}{z + i\epsilon}$ requires these steps:



and

$$\int_{\frown} dz\,\frac{f(z)}{z} = f(0)\int_{\frown}\frac{dz}{z} = f(0)\int_{2\pi}^{\pi} i\,d\theta = -i\pi f(0)$$

whence the rule

$$\frac{1}{x - x_0 \pm i\epsilon} = P\,\frac{1}{x - x_0} \mp i\pi\,\delta(x - x_0).$$

Cauchy's inequality:

Suppose $f(z)$ is analytic in a region that includes the circle

$$z = re^{i\theta},$$

and suppose that on this circle $f(z)$ is bounded by

$$|f(z)| \leq M.$$

Then since (6.47)

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{z^{n+1}}, \quad \text{we have}$$

$$|f^{(n)}(0)| \leq \frac{n!}{2\pi} \oint \frac{|f(z)|\,|dz|}{|z|^{n+1}}$$

$$= \frac{n!}{2\pi} M \int_0^{2\pi} \frac{d\theta}{r^{n+1}} = \frac{n!}{r^{n+1}} M$$