Now $\quad U(g_1)U(g_2) = U(g_1 g_2)$, so if we set

$$D(g)_{ij} = \langle e, i | U(g) | e, j \rangle, \text{ then}$$

$$\sum_k \langle e, i | U(g_1) | e, k \rangle \langle e, k | U(g_2) | e, j \rangle = \langle e, i | U(g_1 g_2) | e, j \rangle$$

or

$$\sum_k D(g_1)_{ik} \, D(g_2)_{kj} = D(g_1 g_2)_{ij} .$$

So the matrices $D(g)$ form a representation of the group

$$D(g_1) \, D(g_2) = D(g_1 g_2).$$

Example: The e.v.'s of H for the non-relativistic H atom are $|n, \ell, m\rangle$ where $\ell = 0, 1, \ldots n-1$ and $m = -\ell, -\ell+1, \ldots \ell-1, \ell$. So there are $2\ell+1$ states for each ℓ and for each n

$$\sum_{\ell=0}^{n-1} 2\ell+1 = n + 2\frac{(n-1)n}{2} = n^2 \text{ states}$$

all with the same energy, a multiplet,

$$e_n = -\frac{1}{2}\frac{mc^2\alpha^2}{n^2} \qquad \alpha = \frac{1}{137.036}.$$

If we include spin, then there are $2n^2$ states all with the same energy.

Without spin, the U's are

$$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{L}}. \qquad (\hbar = 1)$$

With spin, the generators of rotations are

$$\vec{J} = \vec{L} + \vec{S} \qquad \text{and}$$

$$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}}$$

are the unitary operators.

Approximate symmetries also are useful.

$$\begin{pmatrix} u \\ d \end{pmatrix}' = e^{-i\theta\cdot\frac{\sigma}{2}} \begin{pmatrix} u \\ d \end{pmatrix}$$

interchanges u and d quarks. The strong interactions don't care, and the electroweak ones wouldn't care if the Higgs boson were massless.

Infinite Series

Let

$$s_i = \sum_{n=1}^{i} u_n .$$

If for every $\epsilon > 0$, there is an $N(\epsilon)$ such that

$$|s - s_i| < \epsilon \quad \text{if} \quad i > N,$$

then the series $\sum_{n=1}^{\infty} u_n$ is said to <u>converge</u> to the limit s.

A series with partial sums s_i converges if and only if for every $\epsilon > 0$ there is an $N(\epsilon)$ such that

$$|s_j - s_i| < \epsilon \quad \text{when } i > N \text{ and } j > N.$$

This is Cauchy's criterion.

Geometric series. Let

$$s_n = \sum_{i=1}^{n} r^n, \quad \text{where } r \text{ may be complex.}$$

Then $(1-r)s_n = 1 - r^{n+1}$ and so

$$s_n = \frac{1-r^{n+1}}{1-r}.$$

So if $|r|<1$, then $\lim_{n\to\infty} s_n = \frac{1}{1-r}$.

This series converges if $|r|<1$.

Some non-~~convergent~~ series, e.g., $u_n = (-1)^n$ are oscillatory.

Series for which $|s_n| \to \infty$ are said to diverge.

$\sum \frac{1}{n}$ diverges logarithmically

One may test a series to see if it converges, oscillates, or diverges by using one of the several tests explained in sec. 5.2.

Comparison test: If $0 \le u_n \le a_n$ and the series a_n converges, then so does u_n. If the series u_n diverges, then so does a_n.

If $\sum_{n=0}^{\infty} |u_n|$ converges, then

$\sum_{n=0}^{\infty} u_n$ converges absolutely. If $\sum |u_n|$

diverges, but $\sum u_n$ converges, then $\sum u_n$ converges conditionally.

An absolutely convergent series may be reordered; its sum is independent of the order of its terms.

Absolutely convergent series may be multiplied together to form a product, double series that absolutely converges to the product of the two series.

Padé approximations are useful.

Series of functions:

If $\sum_{j=1}^{\infty} u_j(x) = S(x) = \lim_{n\to\infty} s_n(x)$ where

$$s_n(x) = \sum_{j=1}^{n} u_j(x), \quad \text{then}$$

The series of functions $s_n(x)$ converges at x.

If for each $\epsilon > 0$, $\exists$ N such that

$|S(x) - s_n(x)| < \epsilon$ for all $n > N$ and

all $x \in [a,b]$, then the series $s_n(x)$ is <u>uniformly</u> <u>convergent</u> on $[a,b]$.

Weierstrass M test: If ΣM_i converges with $M_i \geq |u_i(x)|$ for all $x \in [a,b]$, then the series $\Sigma\, u_i(x)$ converges uniformly on $[a,b]$.

Taylor's expansion. Assume $f(x)$ has a continuous nth derivative $f^{(n)}(x)$ for $x \in [a,b]$. By integrating $f^{(n)}(x)$ n times, one finds

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2} f''(a) + \cdots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n \qquad \text{where}$$

$$R_n = \int_a^x dx_n \int_a^{x_n} dx_{n-1} \cdots \int_a^{x_2} dx_1 \; f^{(n)}(x_1)$$

$$= \frac{(x-a)^n}{n!} f^{(n)}(\xi) \quad \text{for some} \quad \xi \in [a,b].$$

If $\lim_{n\to\infty} R_n = 0$, then the Taylor series converges

$$\sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a) = f(x)$$

to $f(x)$ on $[a,b]$.

An application of Taylor expansion is the binomial theorem

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m!}{n!\,(m-n)!}\, x^n = \sum_{n=0}^{\infty} \binom{m}{n} x^n$$

which converges for $-1 < x < 1$ and terminates at $n=m$ if m is an integer. If m is not an integer,

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m+1-n)}{n!} .$$

Also $$(a_1+a_2+\cdots+a_m)^n = \sum_{n_1+n_2+\cdots+n_m=n} \frac{n!}{n_1!\,n_2!\cdots n_m!}\, a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m}$$

Translations

$$f(r+a) = \sum_{n=0}^{\infty} \frac{1}{n!} (a\cdot\nabla)^n f(r)$$

$$= e^{a\cdot\nabla} f(r) ,$$

The series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$ converges for all complex x.

$$\psi(\vec{x}+\vec{a}) = e^{i\vec{a}\cdot\vec{p}/\hbar}\, \psi(\vec{x})$$ where $\vec{p} = \hbar \frac{\vec{\nabla}}{i}$.

Power series

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

By the root or ratio test (Sec. 5.2), if

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = R^{-1},$$

then the series converges for $-R < x < R$ and it converges uniformly and absolutely on any closed subinterval $-R < -S \le x \le S < R$. Because $u_n(x) = a_n x^n$ is continuous, the uniform convergence on $[-S, S]$ implies that $f(x)$ is continuous there.

One may differentiate $f(x)$ or integrate it; these power series have the same radius of convergence.

The power-series representation of a function is <u>unique</u>.

Elliptic integral of the first kind:

$$F(\phi\backslash\alpha) = \int_0^{\phi} (1 - \sin^2\alpha \ \sin^2\theta)^{-1/2} \, d\theta$$

or

$$F(x|m) = \int_0^{x} [(1-t^2)(1-mt^2)]^{-1/2} \, dt \qquad 0 \le m < 1.$$

Complete elliptic integral of the first kind:

$$K(m) = \int_0^{\pi/2} (1 - m \sin^2\theta)^{-1/2} \, d\theta$$

$$= \int_0^1 [(1-t^2)(1-mt^2)]^{-1/2} \, dt$$

with $m = \sin^2\alpha$ and $0 \le m < 1.$

Elliptic integral of the second kind:

$$E(\phi\backslash\alpha) = \int_0^{\phi} (1 - \sin^2\alpha \ \sin^2\theta)^{1/2} \, d\theta$$

or

$$E(x|m) = \int_0^{x} \left(\frac{1-mt^2}{1-t^2}\right)^{1/2} dt \qquad 0 \le m \le 1.$$

Complete elliptic integral of second kind:

$$E(m) = \int_0^{\pi/2} (1 - m \sin^2\theta)^{1/2} d\theta = \int_0^1 \left(\frac{1-mt^2}{1-t^2}\right)^{1/2} dt \quad \text{for } 0 \le m \le 1.$$

Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}$$

Bernoulli functions:

$$\frac{x e^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(s) \frac{x^n}{n!}$$

Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s > 1.$$

$$= \prod_{\substack{p=2 \\ p \text{ prime}}}^{\infty} \frac{1}{1 - p^{-s}}$$

Exponential integrals

$$Ei(x) = \int_{-\infty}^{x} \frac{e^u}{u} \, du$$

$$E_1(x) = \int_{x}^{\infty} \frac{e^{-u}}{u} \, du = -Ei(-x)$$

Cosine & Sine integrals

$$Ci(x) = -\int_{x}^{\infty} \frac{\cos t}{t} \, dt \qquad si(x) = -\int_{x}^{\infty} \frac{\sin t}{t} \, dt$$

CV

$$z = (x, y)$$

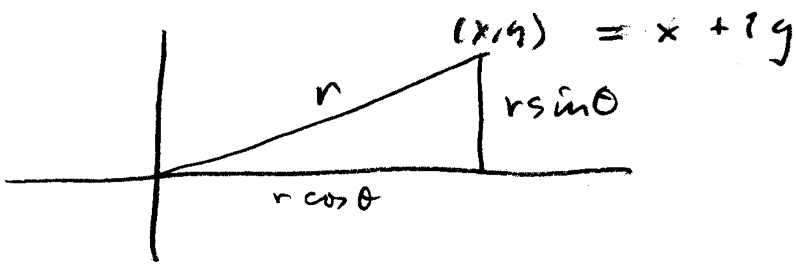
$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

$$z_1 z_2 = (x_1, y_1)\cdot(x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)$$

Let $i = (0,1)$. Then $i^2 = (0,1)\cdot(0,1) = (-1, 0) = -1$, ie, $i = \sqrt{-1}$. Then with $x = x(1,0) = (x,0)$.

$$z = (x,y) = x(1,0) + y(0,1) = x + (0,1)\cdot(y,0) = x + iy .$$

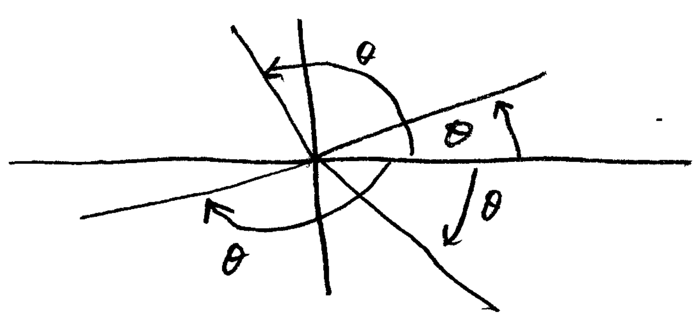
Let $x = r\cos\theta$ & $y = r\sin\theta$



$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Recall $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ & $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.

$r = |z| = (x^2+y^2)^{1/2}$ $\qquad \theta = \tan^{-1}(y/x)$.



$\bar{z} = z^* = x - iy = (x, -y)$

$z\bar{z} = zz^* = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

So $\quad r = \sqrt{z\bar{z}} = |z|.$

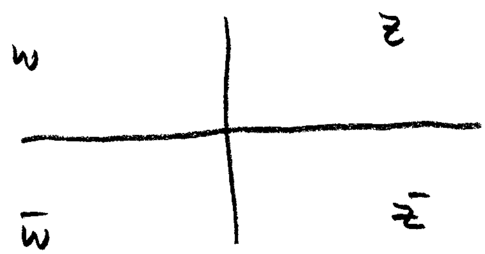
$\triangle$ inequalities:

$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$

$z_1 \cdot z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

So $\quad |z_1 z_2| = |z_1| |z_2|$

$\arg z_1 z_2 = \arg z_1 + \arg z_2.$



$\ln z = \ln re^{i\theta} = \ln r + i\theta$ but

$\ln z = \ln re^{i(\theta + 2\pi n)} = \ln r + i(\theta + 2n\pi).$

Cauchy - Riemann conditions

When does

$f(x,y) = u(x,y) + i\,v(x,y)$ have a derivative

w.r.t. $z = x + iy$?

$$\frac{df}{dz} = \frac{u_x\,dx + u_y\,dy + i\,v_x\,dx + i\,v_y\,dy}{dx + i\,dy}$$

$$\frac{df}{dx} = \frac{u_x\,dx + i\,v_x\,dx}{dx} = u_x + i\,v_x$$

$$\frac{df}{i\,dy} = \frac{u_y\,dy + i\,v_y\,dy}{i\,dy} = -i\,u_y + v_y$$

So $\frac{df}{dx} = \frac{df}{i\,dy}$ only if $u_x + i\,v_x = v_y - i\,u_y$

i.e.

$$u_x = v_y \quad \text{and} \quad v_x = -u_y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

When these C-R conditions apply,

$$\frac{df}{dz} = \frac{u_x\,(dx + i\,dy) + i\,v_x\,(dx + i\,dy)}{dx + i\,dy} = u_x + i\,v_x .$$

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial iy}$$

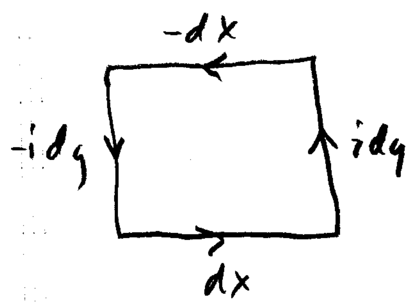
$f(z)$ is <u>analytic</u> at $z = z_0$ if it is differentiable at $z = z_0$ and in a small region around z_0.

If $f(z)$ is analytic for all z, then it is <u>entire</u>.

Contour integrals.

$$\int f(z)\,dz = \lim_{N \to \infty} \sum_{i=1}^{N-1} f(z_i)\,(z_{i+1} - z_i)$$

Say $f(z)$ is analytic in a region and so satisfies the C-R conditions there.



$$\oint f(z)\,dz = (u + iv)\,dx - (u + iv)\,i\,dy$$

$$+ (u + u_x dx + iv + iv_x dx)\,i\,dy$$

$$- (u + u_y dy + iv + iv_y dy)\,dx$$ so

$$\oint f(z)\,dz = (i u_x - v_x - u_y - i v_y)\,dx\,dy = 0$$ if

$u_x = v_y$ and $-v_x = u_y$.

In general

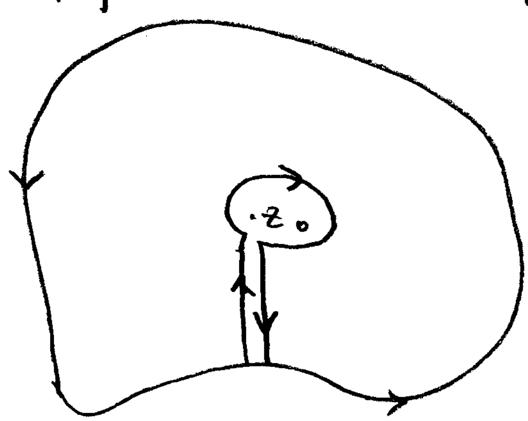
$$\oint f(z)\,dz = 0$$

as long as the contour lies within a region in which $f(z)$ is analytic and in which one could shrink the contour to a point.

Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz$$

where z_0 and the contour are within a (simply connected) region of analyticity.



$$\frac{1}{2\pi i} \oint \frac{f(z)\,dz}{z - z_0} = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)\,dz}{z - z_0} + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)\,dz}{z - z_0}$$