Examples of the calculus of residues:

Consider the integral

$$I(t,m) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k^2+m^2} .$$

First assume $t>0$. Then we may close the contour by adding a hemispheric contour

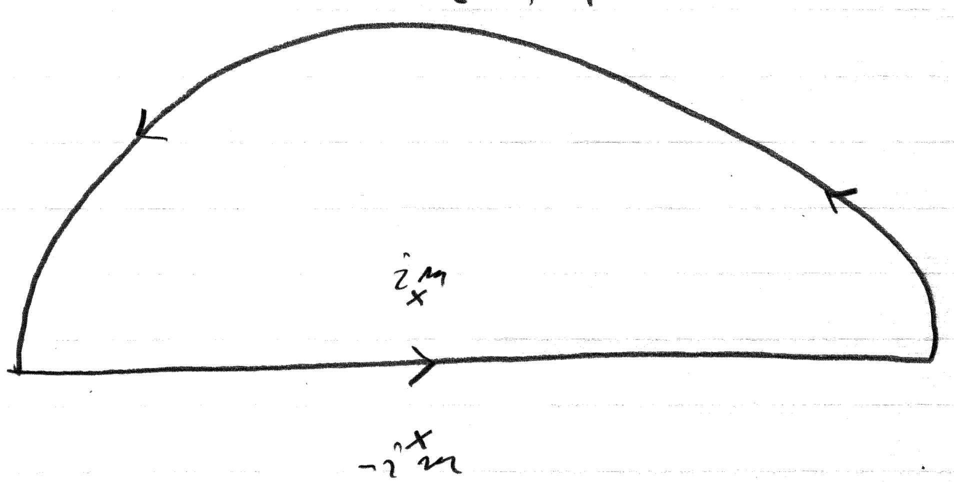
$$z = R e^{i\theta} \qquad 0 \le \theta \le \pi$$

and letting $R \to \infty$. Now for $k = z = Re^{i\theta}$,

$$e^{ikt} = e^{iRt(\cos\theta + i\sin\theta)} = e^{iRt\cos\theta} \, e^{-Rt\sin\theta} .$$

So for $t>0$, $e^{ikt} \to 0$ as $R \to \infty$. So by adding the extra hemispheric contour, we do not change $I(t,m)$:

$$I(t,m) = \oint dz \, \frac{e^{izt}}{z^2+m^2}$$



There is one pole within the contour because

$$\frac{e^{izt}}{z^2+m^2} = \frac{e^{izt}}{(z-im)(z+im)} .$$

So we may shrink the contour to a tiny circle around $z = im$:

$$I(t,m) = \oint dz \, \frac{e^{izt}}{(z+im)(z-im)}$$

$$= 2\pi i \left. \frac{e^{izt}}{z+im} \right|_{z=im}$$

$$= \frac{2\pi i}{2im} e^{i\,imt} = \frac{\pi}{m} e^{-mt} .$$

What if $t < 0$? Then we may close the contour by adding a lower hemispheric contour

$$k = z = R e^{i\theta} \qquad \pi \le \theta \le 2\pi .$$

The integral

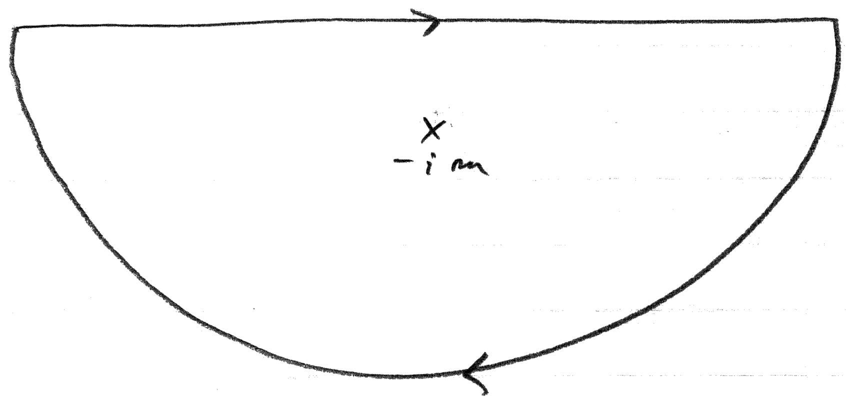
$$\int_{\smile} dz \, \frac{e^{izt}}{z^2+m^2}$$

is zero because for $t<0$

$$e^{izt} = e^{iRt(\cos\theta + i\sin\theta)} = e^{iRt\cos\theta}\, e^{-R|t|\sin\theta|}$$

vanishes as $R \to \infty$. So

$$I(t,m) = \oint \frac{e^{izt}}{(z-im)(z-(-im))}$$



is a huge closed contour, which enclosed one pole. But this contour is clockwise, so

$$I(t,m) = -2\pi i \left(\frac{e^{izt}}{z-im} \right) \Bigg|_{z=-im}$$

$$= \frac{-2\pi i}{-2im} e^{+mt} = \frac{\pi}{m} e^{+mt} \quad \text{for } t<0.$$

Combining the two results, for $t>0$ and $t<0$, we get

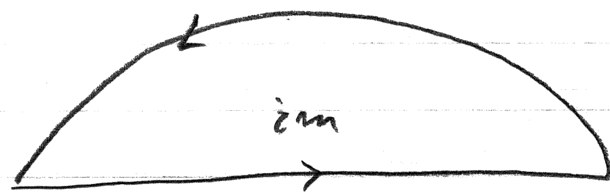
$$\frac{\pi}{m} e^{-m|t|} = \int_{-\infty}^{\infty} dk \frac{e^{ikt}}{k^2+m^2}.$$

As a second example, let's consider the integral

$$J(t,m) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{(k^2+m^2)^2} = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{(k-im)^2(k-(-im))^2} .$$

So for $t>0$, we add the "northern" contour

$$J(t,m) = \oint dz \, \frac{e^{izt}}{(z-im)^2(z-(-im))^2}$$



$$J(t,m) = \oint dz \, \frac{e^{izt}}{(z+im)^2(z-im)^2}$$

$$= 2\pi i \left[\frac{d}{dz} \frac{e^{izt}}{(z+im)^2} \right] \Bigg|_{z=im}$$

where we used Eq. (6.47) for $n=1$. So for $t>0$,

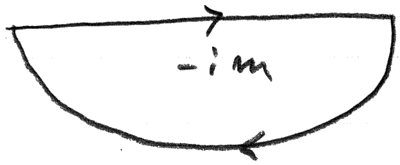
$$J(t,m) = 2\pi i \left[\frac{ite^{-mt}}{(2im)^2} - \frac{2e^{-mt}}{(2im)^3} \right]$$

$$= 2\pi i \, e^{-mt} \left(-\frac{it}{4m^2} - \frac{2i}{8m^3} \right)$$

$$J(t,m) = \frac{2\pi e^{-mt}}{4m^2}\left(t + \frac{1}{m}\right)$$

$$= \frac{\pi}{2m^2}\left(t + \frac{1}{m}\right) e^{-mt} \qquad \text{for } t > 0.$$

For $t < 0$, we add a "southern" contour



So for $t < 0$,

$$J(t,m) = \oint dz \frac{e^{izt}}{(z-im)^2(z-(-im))^2}$$

$$= -2\pi i \left[\frac{d}{dz}\frac{e^{izt}}{(z-im)^2}\right]\Bigg|_{z=-im}$$

$$= -2\pi i \left[\frac{it\, e^{mt}}{(-2im)^2} - \frac{2\, e^{mt}}{(-2im)^3}\right]$$

$$= -\frac{\pi i e^{mt}}{2m^2}\left(-it + \frac{i}{m}\right) = \frac{\pi}{2m^2}\left(\frac{1}{m} - t\right)e^{mt}$$

for $t < 0$.

Combining the two cases, $t > 0$ & $t < 0$, we get

$$\int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{(k^2+m^2)^2} = \frac{\pi}{2m^2}\left(|t| + \frac{1}{m}\right) e^{-m|t|}$$

where $m > 0$.

What if there is a simple pole on the real axis? For instance, consider

$$L(t,m) = \int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)}$$

which is the limit as $\epsilon \to 0$:

$$L(t,m) = \int_{-\infty}^{-\epsilon} dk \, \frac{e^{ikt}}{k(k-im)} + \int_{\epsilon}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)}.$$

Now we'll want to make this a closed contour. So for $t > 0$, we add the "northern" loop,

but we still have the gap at $k = 0$. We may add and subtract either tiny contour,

 or  .

Let's add and subtract $\frown$:

$$L(t,m) = \oint dz \frac{e^{izt}}{z(z-im)} - \int_{\frown} dz \frac{e^{izt}}{z(z-im)}$$

The pole $\frac{1}{z}$ at $z=0$ is now outside the contour. The tiny contour is

$$\int_{\frown} dz \frac{e^{izt}}{z(z-im)} = \frac{1}{-im} \int_{\frown} \frac{dz}{z}$$

Let $z = \epsilon e^{i\theta}$ $\quad dz = i\epsilon e^{i\theta} d\theta$

$$\int_{\frown} dz \frac{e^{izt}}{z(z-im)} = \frac{i}{m} \int_{\pi}^{0} d\theta \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} = -\frac{1}{m} \int_{\pi}^{0} d\theta = \frac{\pi}{m}$$

So

$$L(t,m) = \oint dz \frac{e^{izt}}{z(z-im)} - \frac{\pi}{m}$$

$$= 2\pi i \frac{e^{-mt}}{im} - \frac{\pi}{m}$$

$$= \frac{2\pi}{m} e^{-mt} - \frac{\pi}{m} = \frac{\pi}{m}\left(2e^{-mt} - 1\right).$$

Suppose we had included the residue at $z = 0$. That would have given

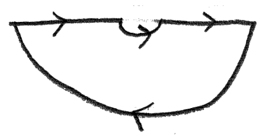
$$2\pi i \, \frac{1}{-im} = -\frac{2\pi}{m} .$$

So we see that the pole on the real axis counts with weight $1/2$.

The addition and subtraction of $\smile$ gives the same result.

Now we do it for $t < 0$:

$$L(t,m) = \oint dz \, \frac{e^{izt}}{z(z-im)} - \int_{\smile} dz \, \frac{e^{izt}}{z(z-im)}$$



$$= 0 - \frac{1}{(-im)} \int_{\pi}^{2\pi} d\theta \, \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}$$

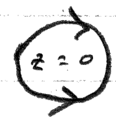
$$= -\frac{i}{m} \, i \int_{\pi}^{2\pi} d\theta = \frac{\pi}{m}$$ which is $1/2$

the residue

$$-2\pi i \, \frac{1}{-im} = +\frac{2\pi}{m}$$

had we included the pole $\frac{1}{z}$ at $z = 0$

by going over it in a clockwise sense



Combining the two results, we have

$$\int_{-\infty}^{\infty} dk \, \frac{e^{ikt}}{k(k-im)} = \begin{cases} \frac{\pi}{m}\left(2e^{-mt} - 1\right) & t > 0 \\ \frac{\pi}{m} & t < 0 \end{cases}.$$

A simple rule for handling simple poles on contours is to go around them both ways and then to average the two answers.