A vector space V is a set of vectors $\{v\}$ and a field F of numbers $\{c\}$ such that if $v_1 \in V$ and $v_2 \in V$ and $c_1 \in F$ and $c_2 \in F$, then

$$v = c_1 v_1 + c_2 v_2 \in V.$$

V is said to be a vector space over F.

3-D Space with $F = R$ is an example.

A set of vectors $e_1, e_2 \cdots$ forms a basis for a vector space V if any vector $v \in V$ can be written as

$$v = \sum_{i=1}^{N} c_i e_i \quad , \qquad c_i \in F.$$

The vectors $\{e_i \mid i=1-N\}$ are complete.

Linear dependence: A set of N vectors v_i for which

$$0 = \sum_{i=1} c_i v_i \qquad (c_i \in F)$$

with some $c_i \neq 0$ is said to be linearly dependent.

If no set of c_i can make the v_i's add to zero

$$0 = \sum_{i=1}^{N} c_i v_i \qquad (\text{except all } c_i = 0),$$

then the vectors v_i are linearly independent.

A basis that is linearly independent is nice because it has no extra, unneeded vectors.

The space R^3 is a vector space. Every triplet of real numbers

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{is a vector in } R^3.$$

The 3 vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

form a basis for R^3 since every $\vec{r}$ can be written as

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= x e_1 + y e_2 + z e_3.$$

R^3 and more generally R^n has a natural scalar product. If $v_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$, then

$$v_1 \cdot v_2 = (v_1, v_2) = \langle v_1 | v_2 \rangle = (x_1, y_1, z_1) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$= x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Another example of a vector space with a scalar product is C^2. A vector $v \in C^2$ is a pair of complex numbers

$$v = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Here $z_i = x_i + i y_i$ and $i^2 = -1$.
Now the natural scalar product for two vectors

$$u = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{is}$$

$$u^\dagger v = (\bar{z}_1, \bar{z}_2) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \bar{z}_1 w_1 + \bar{z}_2 w_2 = \langle u | v \rangle = (u, v),$$

where

$$\bar{z}_i = x_i - i y_i \quad \text{etc.}$$

A nice basis for C^2 is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $v = z_1 e_1 + z_2 e_2$.

Most of the vector spaces of this course have natural scalar products. In all cases, the scalar product or inner product

$(a, b) = \langle a | b \rangle = a^\dagger b$ is linear
$(a, c_1 b_1 + c_2 b_2) = c_1 (a, b_1) + c_2 (a, b_2)$

and antilinear $(c_1 a_1 + c_2 a_2, b) = \bar{c}_1 (a_1, b) + \bar{c}_2 (a_2 b)$ and satisfies

$(a,a) \geq 0$ and $(a,a) = 0$ iff $a = 0$.

Also

$(a,b) = (b,a)^*$ or if the vector space is

real then $(a,b) = (b,a)$. Also if $c, d \in F$, then $(ca, db) = \bar{c} d (a,b)$.

The bases we just considered for R^3 and C^2 have two special properties: they are sets of vectors e_1, e_2, e_3 and e_1, e_2 that are orthogonal and normalized to unity

$(e_i, e_j) = \langle e_i | e_j \rangle = e_i^{\dagger} e_j = \delta_{ij}$ where

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

A set of orthogonal, normalized vectors is <u>orthonormal</u>. Nice bases are orthonormal, as well as complete.

An orthonormal basis $\{e_i \mid i = 1 - N\}$ defines the scalar product for the vector space. For if

$$a = \sum_{i=1}^{N} a_i e_i \quad \text{and} \quad b = \sum_{i=1}^{N} b_i e_i,$$

then

$$(a,b) = \langle a | b \rangle = \sum_{i=1}^{N} \bar{a}_i \sum_{j=1}^{N} b_j (e_i, e_j)$$

$$= \sum_{i,j=1}^{N} \bar{a}_i b_j \delta_{ij} = \sum_{i=1}^{N} \bar{a}_i b_i .$$

Any set $\{v_i \mid i = 1-N\}$ of basis vectors can be made orthonormal (ON). In fact, any set of linearly independent vectors can be made ON.

For instance, the vectors

$$s_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are not ON since $s_1 \cdot s_2 = 1$. We take

$$e_1 = s_1 \quad \text{and} \quad e_2 = a s_2 + b e_1. \quad \text{Then}$$

we set

$$0 = e_1 \cdot e_2 = a\, s_1 \cdot s_2 + b\, e_1 \cdot e_1 = a + b. \quad \text{So}$$

we set $b = -a$. Then we normalize

$$e_2 = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} - a \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by setting $a = 1$. So our new ON basis is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Homework problem. Make an ON basis out of

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We usually will work with ON bases.

Suppose $\{e_i \mid i = 1 - N\}$ is an ON basis for a vector space V. Then any $v \in V$ is of the form

$$v = \sum_{i=1}^{N} c_i e_i .$$

What are the c_i's? Well,

$$(e_j, v) = (e_j, \sum_{i=1}^{N} c_i e_i) = \sum_{i=1}^{N} c_i (e_j, e_i)$$

$$= \sum_{i=1}^{N} c_i \delta_{ji} = c_j .$$

So

$$c_j = (e_j, v) = \langle e_j | v \rangle = e_j^{\dagger} \cdot v .$$

Here Dirac's notation is nice:

$$v = \sum_{i=1}^{N} c_i e_i = \sum_{i=1}^{N} c_i |e_i\rangle$$

$$= \sum_{i=1}^{N} \langle e_i | v \rangle |e_i\rangle$$

$$= \sum_{i=1}^{N} |e_i\rangle\langle e_i | v \rangle .$$

Since v is arbitrary, the sum

$$\sum_{i=1}^{N} |e_i\rangle\langle e_i| = 1$$ is the identity operator on V.

Example

$$e_1 = |e_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad e_2 = |e_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then

$$|e_1\rangle\langle e_1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$|e_2\rangle\langle e_2| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} .$$

The relation

$$\sum_{i=1}^{N} |e_i\rangle\langle e_i| = \mathbb{1}$$

is called a completeness relation or a resolution of the identity.

What if there are two ON bases

$\{e_i \mid i = 1-N\}$ and $\{s_i \mid i = 1-N\}$?

Then we have two ways of writing 1:

$$1 = \sum_{i=1}^{N} |e_i\rangle\langle e_i| = \sum_{i=1}^{N} |s_i\rangle\langle s_i|.$$

Then

$$\delta_{nm} = \langle s_n | s_m\rangle = \langle s_n | 1 | s_m\rangle$$

$$= \langle s_n | \sum_{i=1}^{N} |e_i\rangle\langle e_i| s_m\rangle$$

$$= \sum_{i=1}^{N} \langle s_n | e_i\rangle\langle e_i | s_m\rangle$$

$$= \sum_{i=1}^{N} \langle e_i | s_n\rangle^* \langle e_i | s_m\rangle$$

Let $U_{im} = \langle e_i | s_m\rangle$. Then

$$(U^\dagger)_{ni} = U^*_{in} = \langle e_i | s_n\rangle^* = \langle s_n | e_i\rangle$$

So

$$\delta_{nm} = \sum_{i=1}^{N} (U^\dagger)_{ni} U_{im}.$$

Thus the matrix U is unitary: $U^\dagger U = 1$.

Also

$$\delta_{ij} = \langle e_i | e_j\rangle = \sum_n \langle e_i | s_n\rangle\langle s_n | e_j\rangle = \sum_n U_{in} U^\dagger_{nj}$$

So $UU^\dagger = 1$.

The unitary matrix

$$U_{im} = \langle e_i | s_m \rangle$$

relates the basis $\{e_i\}$ to the basis $\{s_i\}$:

$$|s_m\rangle = 1|s_m\rangle = \sum_{i=1}^{N} |e_i\rangle\langle e_i|s_m\rangle$$

$$= \sum_{i=1}^{n} |e_i\rangle U_{im} .$$

The adjoint matrix U^+ goes the other way:

$$|e_i\rangle = 1|e_i\rangle = \sum_{m=1}^{N} |s_m\rangle\langle s_m|e_i\rangle$$

$$= \sum_{m=1}^{N} |s_m\rangle U^+_{mi} .$$

If the two bases are real, so that

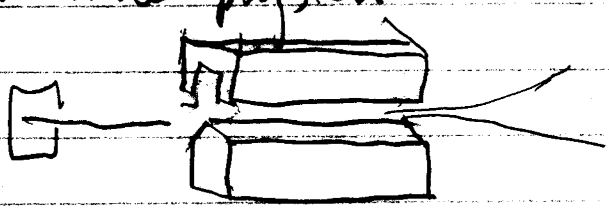
$\langle e_i|s_m\rangle = \langle e_i|s_m\rangle^* = \langle s_m|e_i\rangle$, then

the matrix U is both real and unitary.
In this case, $U^+ = U^T$, where T means transposed.
Then
$1 = U^+U = UU^+ = U^TU = UU^T$ and

$U^{-1} = U^+ = U^T$, so U is orthogonal.

Why are complex bases needed?

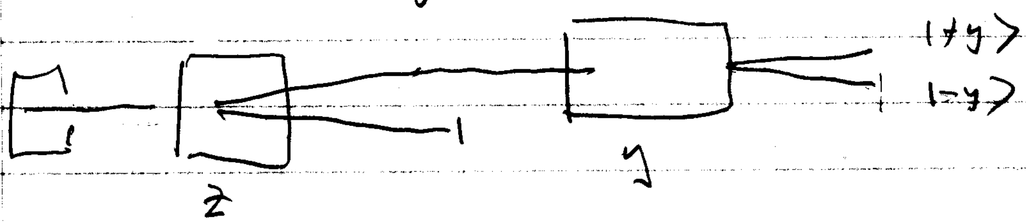
A little physics:

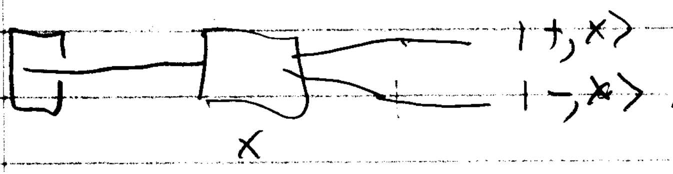


spin up $|+\rangle$

down $|-\rangle$

If this were the whole story, then real vectors would be enough. But





$$e^{-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} = \cos\frac{\theta}{2} - i\,\hat{\theta}\cdot\vec{\sigma}\,\sin\frac{\theta}{2}$$

$$|+y\rangle = e^{-i\frac{\pi}{2}\frac{\sigma_y}{2}}|+\rangle = \left(\cos\frac{\pi}{4} - i\,\sigma_2\sin\frac{\pi}{4}\right)|+\rangle$$

$$= \left(\frac{1}{\sqrt{2}} - i\,\sigma_2\frac{1}{\sqrt{2}}\right)|+\rangle = \frac{1}{\sqrt{2}}\left(1 - i\sigma_2\right)|+\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\begin{pmatrix}0&-i\\i&0\end{pmatrix}\right)|+\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$|-y\rangle = e^{-i\frac{\pi}{2}\frac{\sigma_y}{2}}|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(-|+\rangle + |-\rangle\right)$$

$$|+y\rangle = e^{-i\frac{\pi}{2}\left(-\frac{\sigma_x}{2}\right)}|+\rangle = \cos\frac{\pi}{4} + i\,\sigma_x \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}\left(1 + i\,\sigma_x\right)|+\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1&0\\0&1\end{pmatrix} + i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)|+\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+\rangle + i|-\rangle\right)$$

$$|-y\rangle = e^{-i\frac{\pi}{2}\frac{\sigma_x}{2}}|+\rangle = \frac{1}{\sqrt{2}}\left(1 - i\,\sigma_x\right)|+\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} - i\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]|+\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1&-i\\-i&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|+\rangle - i|-\rangle\right)$$

So the e-states are

$$|\pm z\rangle = |\pm\rangle$$

$$|\pm x\rangle = \frac{1}{\sqrt{2}}\left(\pm|+\rangle \mp |-\rangle\right)$$

$$|\pm y\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right)$$

How could we have all these states using only real numbers?