And

$$\frac{1}{c}\frac{\partial \nabla\times E}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = \nabla\times\frac{1}{c}\frac{\partial E}{\partial t} = \nabla\times(\nabla\times B)$$

$$= -\Delta B \quad \text{since } \nabla\cdot B = 0$$

So

$$\frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = \Delta B \quad \& \quad \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \Delta E.$$

Line integral.

$$\int_C \phi(x)\,d\vec{x} = \sum_i \int \phi(\vec{x})\, e_i\, dx_i$$

$$= \sum_i \vec{e}_i \int_C \phi(\vec{x})\, dx_i$$

$$W = \int \vec{F}\cdot d\vec{r} = \int \sum F_i e_i \cdot \sum e_j\, dx_j$$

$$= \int \sum \delta_{ij}\, F_i\, dx_j = \sum_i \int F_i\, dx_i$$

Surface

$$\int \phi\, d\vec{\sigma}, \quad \int \vec{V}\cdot d\vec{\sigma}, \quad \int \vec{V}\times d\vec{\sigma}$$

$$\int \vec{E}\cdot d\vec{\sigma} = \sum_i \int E_i(x)\, e_i \sum e_j\, d\sigma_j = \int \sum_i E_i\, d\sigma_i$$

$$= \int \nabla\cdot E\, dV = \int 4\pi\rho\, dV = 4\pi Q.$$

$$\int B \cdot d\sigma = \int \nabla \cdot B \, dv = 0$$

$$\int \phi(x) \, d^3x \quad , \qquad \int \vec{V}(\vec{x}) \, d^3x = \sum_i e_i \int V_i(x) \, d^3x .$$

Integral Definitions of ∇, $\nabla\cdot$, $\nabla\times$

Book uses rectangular coordinates.
Let's use spherical ones.

$$\frac{\int \phi \, d\vec{\sigma}}{\int d\tau} = \nabla\phi \quad \text{is what we'll show.}$$

Well,

$$\int \phi(x)\, d\vec{\sigma} = \int\left[\phi(x_0) + \vec{\nabla}\phi \cdot (\overrightarrow{x - x_0})\right] d\vec{\sigma}$$

$$= \int \left[\phi(x_0) + \overrightarrow{\nabla\phi} \cdot (\overrightarrow{x - x_0})\right] (\overrightarrow{x - x_0})\, |x - x_0|\, d\Omega$$

Now

$$\int \phi(x_0)\, (\overrightarrow{x - x_0})\, |x - x_0|\, d\Omega = 0$$

$$\int \overrightarrow{\nabla\phi} \cdot \vec{r}\ \vec{r}\ |r|\, d\Omega = r^3 \int \nabla\phi \cdot \hat{r}\ \hat{r}\, d\Omega$$

$$= r^3 \int \Big(\nabla\phi_x \cos^2\phi \sin^2\theta \, d\Omega,$$
$$\nabla\phi_y \sin^2\phi \sin^2\theta \, d\Omega, \ \nabla\phi_z \cos^2\theta \, d\Omega \Big)$$

$$\int \cos^2\phi \sin^2\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\theta = \frac{2\pi}{2}\int_{-1}^{1} dx\, (1 - x^2) = \frac{4\pi}{3}$$

$$\int \sin^2\phi \sin^2\theta \, d\Omega = \frac{4\pi}{3}$$

$$\int \cos^2\theta \, d\Omega = 2\pi \int_{-1}^{1} x^2 dx = 2\pi \left. \frac{x^3}{3} \right]_{-1}^{1} = 2\pi \frac{2}{3} = \frac{4\pi}{3}$$

So

$$\int \phi \, d\vec{\sigma} = \frac{4\pi}{3} r^3 \vec{\nabla}\phi$$

$$\int d\tau = \frac{4\pi}{3} r^3$$

So $$\vec{\nabla}\phi = \frac{\int \phi \, d\vec{\sigma}}{\int d\tau} .$$

We worked out $\int V \cdot d\sigma$. Let's do it again:

$$\vec{V}(x) = \vec{V}(x_0) + \vec{e}_i \frac{\partial V_i}{\partial x_j} (x - x_0)_j \quad , \qquad \text{so}$$

$$\vec{d\sigma} = \hat{r} \, r^2 d\Omega, \qquad \text{so}$$

$$\int \vec{V} \cdot \vec{d\sigma} = \int V(x_0) \, \hat{r} \, r^2 d\Omega + \int \frac{\partial V_i}{\partial x_j} r_j \, e_i \cdot \hat{r} \, r^2 d\Omega$$

$$= \frac{\partial V_i}{\partial x_j} r^3 \int \hat{r}_i \hat{r}_j \, d\Omega$$

Now $\int \frac{xy}{r^2} d\Omega = 0$

$$\int \hat{r}_i \hat{r}_j \, d\Omega = \frac{4\pi}{3} \delta_{ij}$$

$$\int V \cdot d\sigma = \frac{4\pi r^3}{3} \sum_{ij} \frac{\partial v_i}{\partial x_j} \delta_{ij} = \frac{4\pi r^3}{3} \sum \frac{\partial v_i}{\partial x_i}$$

So since

$$\int d\tau = \frac{4\pi r^3}{3},$$

$$\nabla \cdot V = \sum_i \frac{\partial v_i}{\partial x_i} = \frac{\int \vec{V} \cdot d\vec{\sigma}}{\int d\tau}$$

in the limit in which the sphere is very tiny.

$$\int d\sigma \times V = \sum_{ijk} \int e_i \, \epsilon_{ijk} \, d\sigma_j \, V_k$$

$$d\sigma_j = r v_j \, d\Omega$$

$$V_k(x) = V_k(x_0) + \frac{\partial V_k}{\partial x_\ell} dx_\ell \;; \quad dx_\ell = r_\ell = (x - x_0)_\ell$$

$$\int d\vec{\sigma} \times \vec{v} = \sum \int e_i \, \epsilon_{ijk} \, r \, r_j \, d\Omega \left(V_k + \frac{\partial V_k}{\partial x_\ell} r_\ell \right)$$

$$= \frac{4\pi}{3} r^3 \, \delta_{\ell j} \, e_i \, \epsilon_{ijk} \frac{\partial V_k}{\partial x_\ell}$$

$$= \frac{4\pi}{3} r^3 \; \nabla \times V$$

So

$$\nabla \times V = \frac{\int d\vec{\sigma} \times \vec{v}}{\int d\tau}.$$

Now since

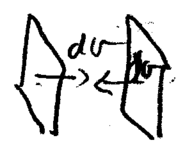
$$\int V \cdot d\sigma = \nabla \cdot V \int d\tau \qquad \text{ie}$$

$$\int V \cdot d\sigma = \int \nabla \cdot V \, d\tau \qquad \text{tiny sphere or cube.}$$

$$\downarrow$$

$$\int V \cdot d\sigma = \int \nabla \cdot V \, d\tau$$

Gauss's theorem.



these cancel as doors close

Gauss's law:

Now $\nabla \cdot E = 4\pi\rho$ So

$$\int E \cdot d\sigma = \int \nabla \cdot E \, d\tau = \int 4\pi\rho \, d\tau = 4\pi Q$$

E.g.,

$$E_i = q \frac{r_i}{r^3}$$

$$\int \vec{E} \cdot \vec{d\sigma} = \int q \frac{r_i}{r^3} r_i r \, d\Omega = \int q \, d\Omega = 4\pi q$$

So

$$= \int \nabla \cdot E \, d\tau = \int 4\pi\rho \, d\tau$$

$$= 4\pi q,$$

$$\int \vec{B} \cdot \vec{d\sigma} = \int \nabla \cdot B \, d\tau = 0.$$

But also for a tiny volume,

$$\int \nabla \times V \, d\tau = \int d\sigma \times V$$ in flate

$$\int \nabla \times V \, d\tau = \int d\sigma \times V.$$ cancellation

Also

$$\int \vec{\nabla} \phi \, d\tau = \int \phi \, \vec{d\sigma}.$$ (from p38)

Green's theorem

$$\nabla\cdot(u\nabla v) = e_i \frac{\partial}{\partial x_i} \cdot u \frac{\partial v}{\partial x_j} e_j$$

$$= \sum_{ij} \delta_{ij} \frac{\partial}{\partial x_i}\left(u \frac{\partial v}{\partial x_j}\right) = \sum_i \frac{\partial}{\partial x_i} u \frac{\partial v}{\partial x_i}$$

$$= \sum_i \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + u \frac{\partial^2}{\partial x_i^2} v \qquad \text{ie}$$

$$\nabla\cdot(u\nabla v) = \vec{\nabla} u \cdot \vec{\nabla} v + u \Delta v$$

$$\nabla\cdot(v\nabla u) = \nabla v \cdot \nabla u + v \Delta u \qquad \text{so}$$

$$\int(u\Delta v - v\Delta u)\,d\tau = \int[\nabla\cdot(u\nabla v) - \nabla\cdot(v\nabla u)]\,d\tau$$

$$= \int(u\vec{\nabla} v - v\vec{\nabla} u)\cdot d\vec{\sigma} \qquad \underline{\text{Green}}.$$

Also

$$\int \nabla\cdot(u\nabla v)\,d\tau = \int u\vec{\nabla} v \cdot d\vec{\sigma}$$

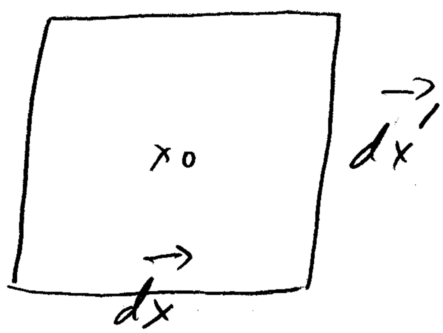
$$= \int(\vec{\nabla} u \cdot \vec{\nabla} v + u\Delta v)\,d\tau$$

We saw that

$$\int_{\text{tiny } x\text{-}y \text{ square}} V \cdot dl = (\nabla \times V)_z \; da = (\nabla \times V) \cdot \vec{d\sigma} \quad \text{in } \hat{z} \text{ dir.}$$

area of square.

$$V_i(x) = V_i(x_0) + \sum_j \frac{\partial V_i}{\partial x_j}(x_j - x_{0j})\frac{1}{2}$$



$$\vec{d\sigma} = \vec{dx} \times \vec{dx}' \qquad \text{so}$$

$$(\nabla \times V) \cdot d\sigma = \sum \vec{e}_i \; \epsilon_{ijk} \frac{\partial V_k}{\partial x_j} \cdot \left(\vec{dx} \times \vec{dx}'\right)$$

$$= \sum \epsilon_{ijk} \frac{\partial V_k}{\partial x_j} \; \epsilon_{irs} \; dx_r \, dx_s'$$

$$= \sum \left(\delta_{jr} \delta_{ks} - \delta_{js} \delta_{kr}\right) \frac{\partial V_k}{\partial x_j} dx_r \, dx_s'$$

$$= \sum_{j,k} \frac{\partial V_k}{\partial x_j} dx_j \, dx_k' - \frac{\partial V_k}{\partial x_j} dx_k \, dx_j'$$

$$\int \vec{V}\cdot d\vec{l} = \left(V_i(x_0) - \frac{\partial V_i}{\partial x_j}\frac{dx'_j}{2}\right)dx_i$$

$$+ \left(V_i(x_0) + \frac{\partial V_i}{\partial x_j}\frac{dx_j}{2} \right) d'x_i$$

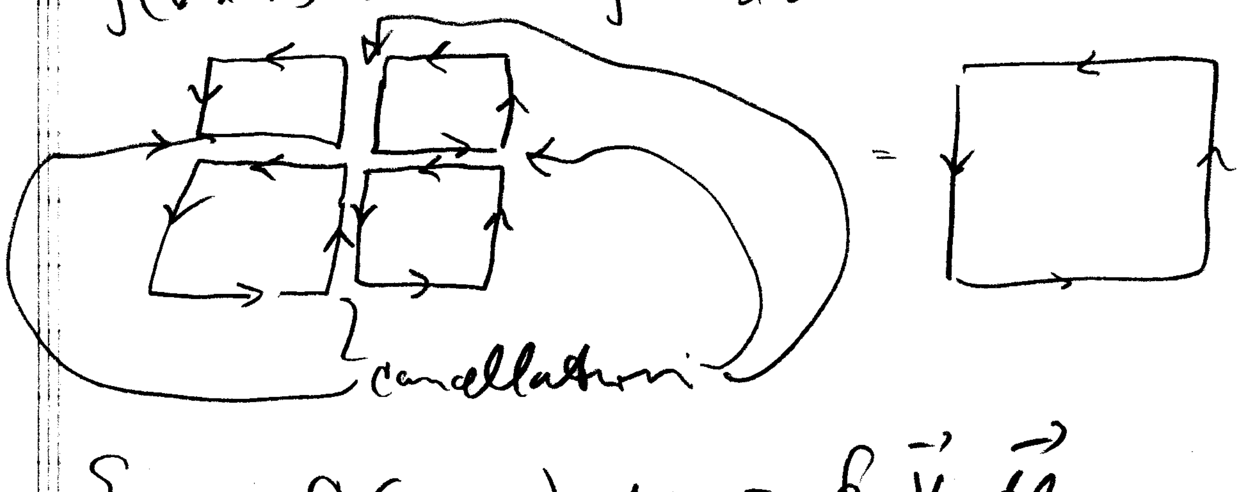
$$- \left(V_i(x_0) + \frac{\partial V_i}{\partial x_j}\frac{dx'_j}{2} \right) dx_i$$

$$- \left(V_i(x_0) - \frac{\partial V_i}{\partial x_j}\frac{dx_j}{2} \right) d'x_i$$

$$= \sum_{ij} \frac{\partial V_i}{\partial x_j} dx_j\, d'x_i - \frac{\partial V_i}{\partial x_j} dx_i\, d'x_j$$

Now then for a tiny area

$$\int (\nabla \times V)\cdot d\sigma = \oint V\cdot dl$$



So $$\int (\nabla \times V)\cdot d\sigma = \oint \vec{V}\cdot d\vec{l}.$$

Stokes's theorem.

Now

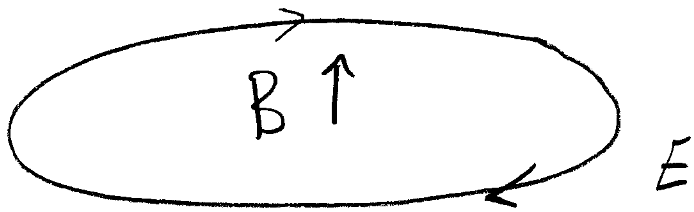
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}.$$

So

$$\int(\vec{\nabla \times E})\cdot d\vec{\sigma} = \oint \vec{E}\cdot d\vec{l}$$

$$\parallel$$

$$-\frac{1}{c}\frac{\partial}{\partial t}\int B\cdot d\sigma$$



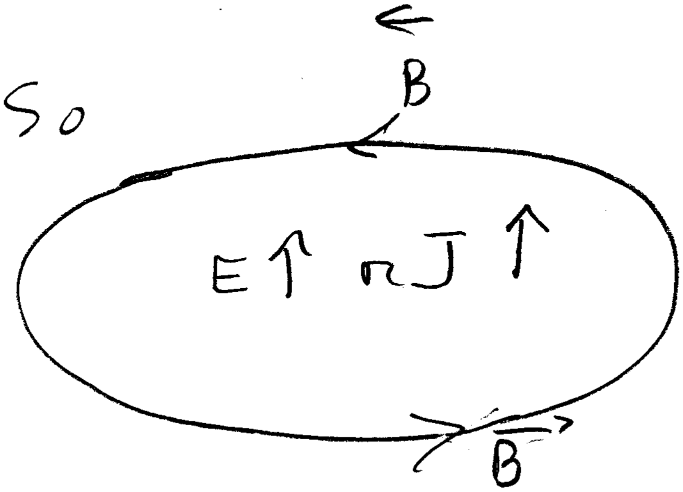
So if B is increasing, then there will be a clockwise E field.

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t} \qquad \text{So}$$

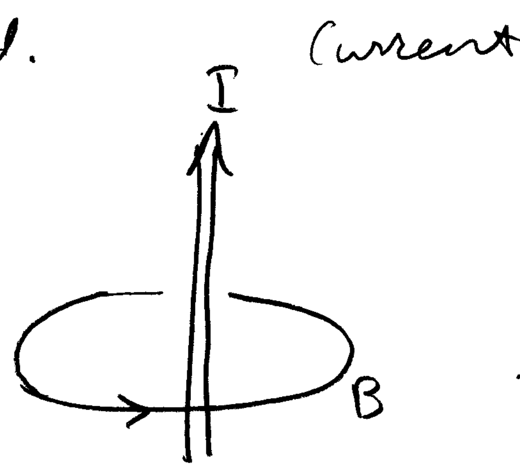
$$\int(\nabla \times B)\cdot d\sigma = \oint B\cdot dl = \int \frac{4\pi}{c} J\cdot d\sigma + \frac{1}{c}\frac{\partial}{\partial t}\int E\cdot d\sigma$$

$$= \frac{4\pi}{c} I + \frac{1}{c}\frac{\partial}{\partial t}\int \vec{E}\cdot d\vec{\sigma}$$

So



So a current or an increasing $\vec{E}$ will cause a counter-clockwise B field.



Now $\int (\nabla \times V) \cdot d\sigma = \oint V \cdot dl$

Let $V = \vec{a}\phi$ $\quad\vec{a}$ a constant.

then

$$\int (\nabla \times \vec{a}\phi) \cdot d\sigma = \int \vec{a}\,\phi \cdot dl = \vec{a} \cdot \int \phi\, d\vec{l}$$

$$\parallel$$

$$\int \epsilon_{ijk} \frac{\partial}{\partial x_j}(a_k\phi)\, d\sigma_i = a_k \int \epsilon_{kij}\, d\sigma_i \frac{\partial \phi}{\partial x_j}$$

$$= \vec{a} \cdot \int d\sigma \times \nabla \phi \qquad \text{so}$$

$$\int d\vec{\sigma} \times \vec{\nabla}\phi = \int \phi\, d\vec{l}.$$

If $V = a \times P$, then

$$\int \nabla \times (a \times P) \cdot d\sigma = \oint (a \times P) \cdot dl$$

$$\parallel \qquad = \oint dl_i\, \epsilon_{ijk} a_j P_k$$

$$\int d\sigma_i\, \epsilon_{ijk} \nabla_j \epsilon_{k\ell m} a_\ell P_m \qquad = a_j \oint \epsilon_{jki} P_k\, dl_i$$

$$\parallel$$

$$-a_\ell \int \epsilon_{\ell k m} \epsilon_{kij}\, d\sigma_i \nabla_j P_m \qquad = \vec{a} \cdot \int P \times dl$$

$$\parallel$$

$$-a_\ell \int \epsilon_{\ell k m} (d\vec{\sigma} \times \vec{\nabla})_k P_m = -\vec{a} \cdot \int (d\sigma \times \nabla) \times P$$

So

$$\int (d\sigma \times D) \times P = \int \vec{dl} \times \vec{P}$$

HW

1.6.4
1.7.6
1.8.7
1.8.8
1.8.9
1.8.15
1.8.16
1.9.12
1.10.4
1.10.5
1.11.7
1.11.9
1.12.5

1.9.13

$$\Delta \psi = h\, \nabla\phi^2$$

$$\Delta\phi = 0$$

let $\psi = \frac{1}{2} h \phi^2$, then

$$\nabla\psi = h\phi\nabla\phi$$

$$\Delta\psi = h\nabla\phi \cdot \nabla\phi = h\, \nabla\phi^2$$

1.6.5 $f(u,v) = 0$

$$0 = \nabla f = f_u \nabla u + f_v \nabla v = 0$$

$$0 = \nabla u \times \nabla f = f_v \nabla u \times \nabla v = 0$$

$$0 = \nabla f \times \nabla v = f_u \nabla u \times \nabla v$$

So $\nabla u \times \nabla v = 0$ unless $f_u = f_v = 0$.

In a simply connected (no holes) region,

$$\vec{F} = -\nabla\phi \iff \vec{\nabla}\times\vec{F} = 0 \iff \oint \vec{F}\cdot d\vec{r} = 0$$

and we may describe the conservative force $\vec{F}$ by the scalar ϕ. We may add a constant to ϕ.

If $F = -\nabla\phi$, then $\nabla\times F = -\nabla\times\nabla\phi = 0$ since

$$\nabla\times F = -\epsilon_{ijk}\nabla_j\nabla_k\phi = 0.$$

If $\vec{\nabla}\times\vec{F} = 0$, then in a simply connected region

$$\oint \vec{F}\cdot d\vec{r} = \int(\nabla\times F)\cdot d\vec{\sigma} = 0.$$

If $\oint \vec{F}\cdot d\vec{r} = 0$, then the definition

$$\phi(\vec{x}) = \phi(\vec{x}_0) - \int_{x_0}^{x} \vec{F}(\vec{x}')\cdot dx'$$

is path independent, since

$$\phi_{p_1}(x) - \phi_{p_2}(x) = -\int_{x_0 p_1}^{x} F\cdot dx + \int_{x_0 p_2}^{x} F\cdot dx$$

$$= \oint_{p_2, p_1} F\cdot dx = 0.$$

So the work is path independent and depends only on the end points. Force is conservative; it conserves energy.

The gravitational and electrostatic potentials are conservative.

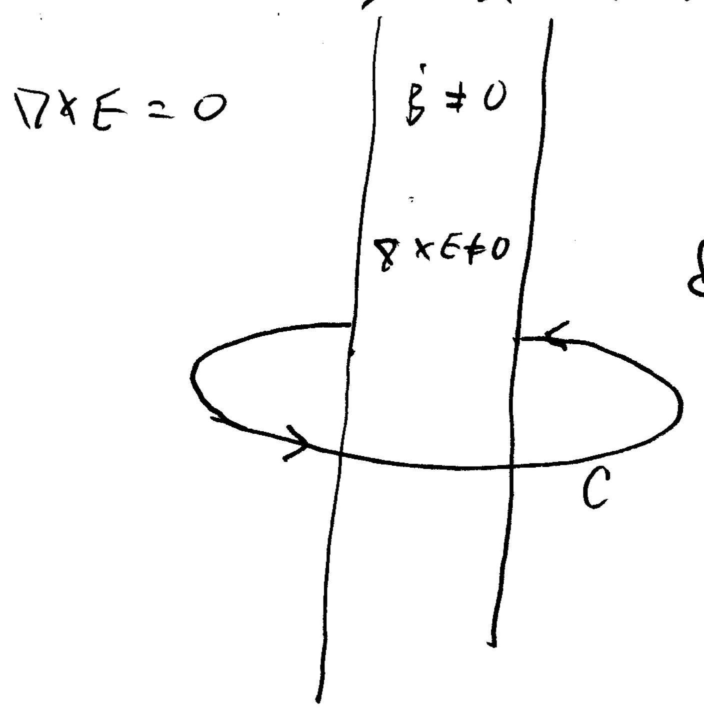
The magnetic force $F = q\, v \times B$ does no work since

$$F \cdot dr = q\, v\, dt \cdot (v \times B) = 0.$$

So as long as $\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} = 0$, we may define a conservative potential.

Why did we exclude holes?

Suppose $\vec{B}$ is time independent everywhere except in a tube, then outside the tube $\nabla \times E = 0$, and we might try to define ϕ



But

$$\oint_C \vec{E} \cdot d\vec{r} = \int \nabla \times E\, d\sigma$$
$$= -\frac{1}{c} \int \dot{\vec{B}} \cdot d\vec{\sigma} \neq 0$$

Exact Differentials

$$df = P(x,y)\,dx + Q(x,y)\,dy$$

is exact if it is path independent, ie if

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \quad \text{ie}$$

$$P(x,y) = \frac{\partial f}{\partial x} \qquad Q(x,y) = \frac{\partial f}{\partial y}$$

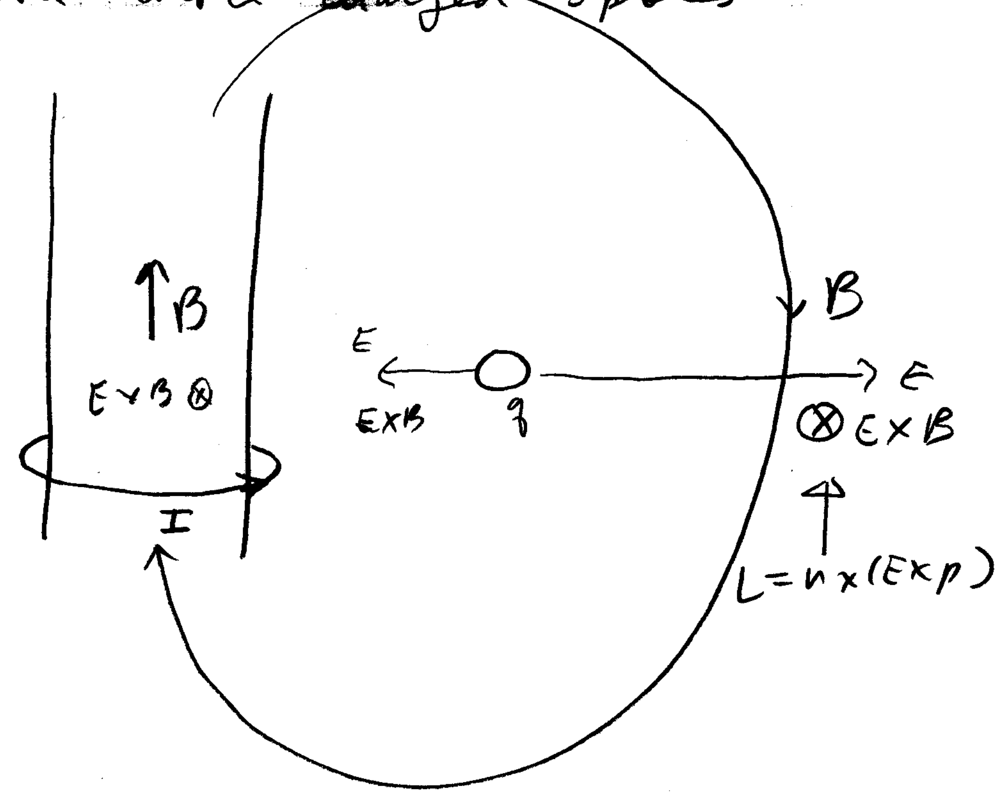
but then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

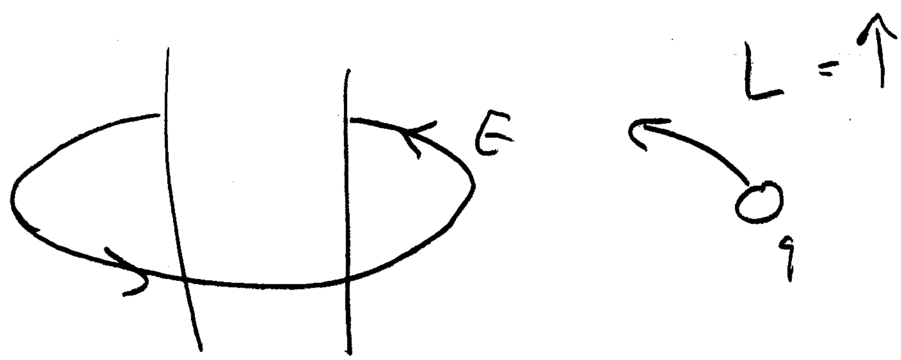
In this case $\vec{F} = (P, Q, 0)$ has

$$(\nabla \times F)_z = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Solenoid with charged spheres

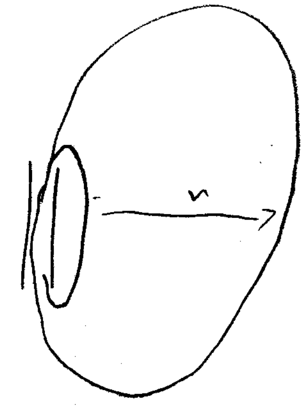


Now cut current $\dot{B} < 0$



$\int_{r_0}^{\infty} r\,dr\, B_r \sim C$ So if $B_r \sim \frac{1}{r^3}$

$\int_{r_0}^{R} \frac{dr}{r^2} = \log \frac{R}{r_0}$ $\oint B\cdot dr = \frac{4\pi}{c} I$



We set $\vec{B} = \vec{\nabla} \times \vec{A}$ so as to satisfy

$0 = \nabla \cdot B = \nabla \cdot (\nabla \times A) = 0$ automatically.

We may choose $\vec{A} = \vec{e}_2 A_2 + \vec{e}_3 A_3$ (by doing a gauge transformation.) Then

$$B_2 = -\frac{\partial A_3}{\partial x_1} + \frac{\partial A_1}{\partial x_3} = -\frac{\partial A_3}{\partial x_1}$$

$$B_3 = +\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = +\frac{\partial A_2}{\partial x_1}$$

So

$$A_2 = \int_{x_{10}}^{x_1} B_3 \, dx_1' + f_2(x_2, x_3)$$

$$A_3 = -\int_{x_{10}}^{x_1} B_2 \, dx_1' + f_3(x_2, x_3).$$

We have B_2 and B_3 and need B_1:

$$B_1 = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = \int -\frac{\partial B_2}{\partial x_2} - \frac{\partial B_3}{\partial x_3} \, dx_1' + \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}$$

$$= \int_{x_{10}}^{x} \frac{\partial B_1}{\partial x_1'} \, dx_1' + \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \qquad (\nabla \cdot B = 0)$$

$$= B_1(x) - B_1(x_0) + \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}$$

Let $f_2 = 0$ $\qquad f_3 = \int_{x_{20}}^{x_2} B_1(x_{10}, x_2', x_3) \, dx_2'$

Any $A' = A + \nabla s$,

where s is any scalar, also works, since

$$\nabla \times A' = \nabla \times A + \nabla \times \nabla s = \nabla \times A = B.$$

It is this gauge freedom that one may use to set $A_1 = 0$

$$\vec{A}' = A + \nabla s$$

$$0 = A_1' = A_1 + \frac{\partial s}{\partial x_1} \qquad \text{so}$$

$$\frac{\partial s}{\partial x_1} = -A_1 \qquad \text{so}$$

$$s(x_1, x_2, x_3) = -\int_{x_{10}}^{x_1} dx_1' \, A_1(x_1', x_2, x_3).$$

$$\vec{A}' = A + \nabla s \quad \text{has} \quad A_1' = 0.$$

For constant B, we may set

$$\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r})$$

$$(\nabla \times A)_i = \sum \tfrac{1}{2} \epsilon_{ijk} \partial_j \epsilon_{k\ell m} B_\ell r_m = \sum \delta_{mj} \tfrac{B_\ell}{2} \epsilon_{ijk} \epsilon_{\ell mk}$$

$$= \sum_{jk} \tfrac{1}{2} \epsilon_{ijk} \epsilon_{\ell jk} B_\ell = \sum \delta_{i\ell} B_\ell = B_i.$$

So under the gauge transformation

$A_\mu \to A_\mu' = A_\mu + \frac{\partial s}{\partial x^\mu}$ the Maxwell tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$$

$$F'_{\mu\nu} = \partial_\mu (A_\nu + \partial_\nu s) - \partial_\nu (A_\mu + \partial_\mu s)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \partial_\nu s - \partial_\nu \partial_\mu s$$

$= F_{\mu\nu}$ is unchanged.

So $\vec{E} \sim F_{0i}$ and $B \sim F_{ij}$ are unchanged.

$D_\mu = \partial_\mu - i\frac{e}{\hbar} A_\mu$ is a covariant derivative

$$D'_\mu \exp(ies/\hbar) = \left[\partial_\mu - i\frac{e}{\hbar}\left(A_\mu + \frac{\partial s}{\partial x^\mu}\right)\right] e^{ies/\hbar}$$

$$= e^{\frac{ies}{\hbar}}\left(\frac{ie\partial s}{\hbar \partial x^\mu} + \partial_\mu - i\frac{e}{\hbar} A_\mu - i\frac{e}{\hbar}\frac{\partial s}{\partial x^\mu}\right)$$

$$= e^{ie\frac{s}{\hbar}} \left(\partial_\mu - \frac{ie}{\hbar} A_\mu\right)$$

$$= e^{ies/\hbar} D_\mu .$$

So the covariant derivative transforms covariantly under a gauge transformation.

This means that if

$$\left[\frac{\hbar}{i} D_\mu^2 + m^2c^2\right]\psi = 0,$$

then also with

$$D'_\mu = \partial_\mu - i\frac{e}{\hbar}A'_\mu = \partial_\mu - \frac{ie}{\hbar}\left(A_\mu + \frac{\partial S}{\partial x^\mu}\right)$$

$$\psi'(x) = e^{\frac{ieS(x)}{\hbar}}\psi(x)$$

then

$$\left[\frac{\hbar}{i} D'^2_\mu + m^2c^2\right]\psi'$$

$$= \left[\frac{\hbar}{i} D'^2_\mu + m^2c^2\right]e^{ieS/\hbar}\psi$$

$$= e^{ieS/\hbar}\left[\frac{\hbar}{i} D_\mu^2 + m^2c^2\right]\psi = 0.$$

There is a non-abelian generalization.

$$[\gamma^\mu D_\mu + m]\psi = 0 \qquad (\hbar = c = 1)$$

Dirac.

Eq. (1.155) is nonsense.

Gauss's Law

$$\vec{E} = \frac{q\hat{r}}{r^2}$$

$$\int \vec{E}\cdot d\vec{s} = \int r^2 d\Omega\, \hat{r}\cdot E = \int d\Omega \frac{r^2 q\, \hat{r}^2}{r^2}$$

small sphere

$$= \int d\Omega\, q = 4\pi q.$$

But over any closed surface that does not enclose the point $\vec{r} = \vec{0}$,

$$\int E\cdot dS = \int \nabla\cdot E\, dv = 0 \text{ since}$$

$$\nabla\cdot\frac{\hat{r}}{r^2} = 0 \quad \text{if} \quad \vec{r} \neq \vec{0}.$$

So

$$\int_V \frac{\nabla\cdot\hat{r}}{4\pi r^2}\, dv = \begin{cases} 1 & \text{if } \vec{0} \in V \\ 0 & \text{if } \vec{0} \notin V. \end{cases}$$

More generally

$$\frac{\nabla \cdot}{4\pi} \frac{\vec{r} - \vec{r}_0}{|r - r_0|^3} = \delta^3(\vec{r} - \vec{r}_0).$$

$$\nabla \cdot \frac{r - r_0}{|r - r_0|^3} = 4\pi\, \delta^3(\vec{r} - \vec{r}_0)$$

by which we mean for smooth f

$$\int f(\vec{r})\ \nabla \cdot \frac{\overrightarrow{r - r_0}}{4\pi |r - r_0|^3}\, dv = f(\vec{r}_0)$$

$$\nabla \cdot E = 4\pi\rho \qquad \text{Gauss's law}$$

This is a constraint. There are no time derivatives of physical quantities here.

If $E = -\nabla\phi$, then

$$\nabla \cdot E = -\nabla \cdot \nabla\phi = -\Delta\phi = 4\pi\rho$$

Poisson
Fish

In vacuum, $\Delta\phi = 0$ Laplace.
the place

$$\vec{F}_E = q_1 q_2 \frac{\overrightarrow{(r_1 - r_2)}}{|r_1 - r_2|^3}$$

$$\vec{F}_G = -G\, m_1 m_2 \frac{\overrightarrow{(r_1 - r_2)}}{|r_1 - r_2|^3}$$

$$\Delta \varphi_G = 4\pi G \rho \,, \qquad \text{for gravity N.R.}$$

Dirac Delta Functional

Let

$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$

or

$$\delta_n(x) = \frac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$$

or $\quad \delta_n(x) = \dfrac{n}{\pi} \dfrac{1}{1 + n^2 x^2} \qquad$ or

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt}\, dt$$

or

$$\delta_n(x) = \frac{1}{2\pi} \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{1}{2}x\right)} .$$

Then if $f(x)$ is smooth

$$\lim_{n\to\infty} \int dx\, f(x)\, \delta_n(x-x_0) = f(x_0).$$

So

$$\int dx\, \delta(x-x_0) \equiv \lim_{n\to\infty} \int dx\, \delta_n(x-x_0)$$

Note $\delta(-x) = \delta(x)$ is even.

$$\int dx\, f(x)\, \delta(a(x-x_0)) = \int dx\, f(x)\, \delta(|a|(x-x_0))$$

$$= \int \frac{|a|dx}{|a|} f(x)\, \delta(|a|(x-x_0))$$

$$= \int \frac{dy}{|a|} f(y/|a|)\, \delta(y-y_0) = f(0)/|a| .$$

So

$$\delta(a(x-x_0)) = \frac{1}{|a|}\delta(x-x_0) ,$$

Now

$$\int f(x)\,\delta(g(x))\,dx = \sum_{\substack{\text{zeros}\\ \text{of } g(x)}} \int f(x)\,\delta\big(g(x_\alpha)+g'(x_\alpha)(x-x_\alpha)\big)\,dx$$

$$= \sum_{\text{zeros}} \int f(x)\,\delta\big(g'(x_\alpha)(x-x_\alpha)\big)\,dx$$

$$= \sum_\alpha \frac{f(x_\alpha)}{|g'(x_\alpha)|} \quad .$$

Suppose we can write

$$f(x) = \sum c_n \phi_n(x) \qquad \text{(completeness)}$$

where $\int \phi_n^*(x)\,\phi_m(x)\,dx = \delta_{nm}$.

Then

$$\int \phi_m^*(x)\,f(x)\,dx = \int \sum c_n\,\phi_m^*(x)\,\phi_n(x)\,dx$$

$$= \sum c_n\,\delta_{nm} = c_m . \qquad \text{So}$$

$$f(x) = \sum c_n\,\phi_n(x) = \sum_n \int \phi_n^*(y)\,f(y)\,\phi_n(x)\,dy$$

But this means that

$$\sum_n \phi_n^*(y)\phi_n(x) = \delta(x-y).$$

closure

A vector field V is uniquely defined in a simply connected region R by its $\nabla\cdot V$ and $\nabla\times V$ and by $V\cdot ds$ on ∂R.
Lemma.

Say $\nabla\cdot V_1 = \nabla\cdot V_2 = s$
$\nabla\times V_1 = \nabla\times V_2 = \vec{c}$

in a simply connected region R on the boundary of which $(V_1 - V_2)\cdot ds = 0$.

Let $\vec{W} = V_1 - V_2$. Then

$$\nabla\cdot W = \nabla\times W = 0.$$

Since $\nabla\times W = 0$, we may set

$$W = -\vec{\nabla}\phi$$

and

$$0 = \nabla\cdot W = -\vec{\nabla}\cdot\vec{\nabla}\phi = -\Delta\phi.$$

Green's theorem (1.104) now says

$$\underbrace{\int_S \phi\nabla\phi\cdot ds}_{0} = \int_V \underbrace{\phi\Delta\phi}_{0}\, dv + \int_V \nabla\phi^2 dv = \int W^2 dv$$

$-\nabla\phi = W = V_1 - V_2$

So $W = 0$ in R
$V_1 = V_2$ in R.

Helmholtz's thm:

If $\vec{V}$, $s = \nabla\cdot V$, and $\vec{c} = \nabla\times V$ vanish as $|\vec{r}|\to\infty$, then

$$\vec{V} = -\vec{\nabla}\phi + \vec{\nabla}\times\vec{A},$$

where

$$\phi(\vec{x}) = \int \frac{s(\vec{y})\,d^3y}{4\pi|\overrightarrow{x-y}|}$$

and

$$\vec{A}(\vec{x}) = \int \frac{\vec{c}(y)\,d^3y}{4\pi|\overrightarrow{x-y}|}.$$

First let's recall that

$$\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{y}}{4\pi|x-y|^3}\right) = -\vec{\nabla}\cdot\nabla\frac{1}{4\pi|x-y|} = \delta^3(\overrightarrow{x-y}).$$

(p. 61, 1.209)

$$\vec{\nabla}\cdot\vec{V} = -\vec{\nabla}\cdot\vec{\nabla}\phi$$

$$= -\vec{\nabla}\cdot\vec{\nabla}\int\frac{s(\vec{y})\,d^3y}{4\pi|x-y|}$$

$$= \int\delta^3(x-y)\,s(y)\,d^3y = s(\vec{x}).$$

Next

$$(\nabla \times V)_i = (\nabla \times (\nabla \times A))_i = \sum \epsilon_{ijk} \nabla_j \epsilon_{klm} \nabla_l A_m$$

$$= \sum (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \nabla_j \nabla_l A_m$$

$$= \nabla_i (\nabla \cdot A) - (\vec{\nabla} \cdot \vec{\nabla}) A_i$$

Now $$\vec{\nabla} \cdot \vec{A} = \int \nabla_i^{(x)} \frac{c_i(y)}{4\pi |x-y|} d^3y$$

$$= -\int c_i(y) \nabla_i^{(y)} \frac{1}{4\pi |x-y|} d^3y$$

$$= \int \frac{1}{4\pi |x-y|} \nabla_i^{(y)} c_i(y) d^3y$$

But $\nabla \cdot c = \nabla \cdot (\nabla \times V) = 0.$ So $\nabla \cdot A = 0.$

And

$$(\nabla \times V) = -\Delta \int \frac{\vec{c}(y) d^3y}{4\pi |x-y|}$$

$$= \int \delta(x-y) c(y) d^3y = \vec{c}(x).$$

So $-\nabla \varphi + \nabla \times A$ have the same div and curl as V in the simply connected region consisting of all of space. And as $|\vec{r}|\to\infty$

V and $-\nabla\phi + \nabla\times A$ both vanish.

Orthogonal Coordinates

$$x_i = x_i(\vec{q}) \qquad q_i = q_i(\vec{x}).$$

$$\vec{e}_i = \frac{\vec{\nabla} q_i}{|\nabla q_i|} \qquad e_i^2 = 1$$

Choose order so that $\quad e_1 \cdot (e_2 \times e_3) > 0.$

$$dx_i = \frac{\partial x_i}{\partial q_j}\, dq_j$$

$$ds^2 = \sum dx_i^2 = \sum_{ijk} \frac{\partial x_i}{\partial q_j} dq_j \frac{\partial x_i}{\partial q_k} dq_k$$

$$= \sum_{jk} g_{jk}\, dq_j\, dq_k$$ where

the metric g is

$$g_{jk} = \sum_i \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k}.$$

For now, we choose orthogonal coordinates

$$e_i \cdot e_j = \delta_{ij}$$

$$g_{ij} = \sum_k \frac{\partial x_k}{\partial q_i} \frac{\partial x_k}{\partial q_j} = \vec{\epsilon}_i \cdot \vec{\epsilon}_j = 0$$

Set $h_i^2 = g_{ii}$ (no sum). Then

$$ds^2 = \sum dx_i^2 = \sum g_{ii} \, dq_i^2$$

$$= \sum_i h_i^2 \, dq_i^2 = \sum_i (h_i dq_i)^2.$$

$\downarrow$ $ds_i = h_i dq_i$ when only one q_i is changed

$$d\vec{r} = \sum_i h_i dq_i \, e_i$$

$$\int v \cdot dv = \sum_i \int v_i h_i dq_i$$

$$d\vec{\sigma}_k = d\vec{\sigma}_{ij} = h_i dq_i \, e_i \times h_j dq_j \, e_j = h_i h_j \, dq_i dq_j \, \epsilon_{ijk} \vec{e}_k$$

cyclic order

NO SUM at all

$$d\tau = d\sigma_1 ds_2 ds_3 = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$$

$$\int \vec{V} \cdot d\vec{\sigma} = \int V_1 h_2 h_3 \, dq_2 \, dq_3$$
$$+ \int V_2 h_3 h_1 \, dq_3 \, dq_1$$
$$+ \int V_3 h_1 h_2 \, dq_1 \, dq_2$$

If $\vec{A} = \sum A_i e_i$ & $\vec{B} = \sum B_j e_j$, then

$$A \cdot B = \sum A_i B_i$$

$$A \times B = \sum A_i B_j \, e_i \times e_j = \sum_{i,j,k} \epsilon_{ijk} e_k A_i B_j$$

$$= \begin{vmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Gradient

$$e_1 \cdot \nabla \psi = \frac{\partial \psi}{\partial s_1} = \frac{1}{h_1} \frac{\partial \psi}{\partial q_1}$$

$$\vec{\nabla} \psi = \sum e_i \frac{1}{h_i} \frac{\partial \psi}{\partial q_i}$$

$$d\psi = \nabla \psi \cdot dr = \sum e_i \frac{1}{h_i} \frac{\partial \psi}{\partial q_i} \cdot \sum e_j h_j \, dq_j$$

$$= \sum \delta_{ij} \frac{h_j}{h_i} \frac{\partial \psi}{\partial q_i} dq_j = \sum \frac{\partial \psi}{\partial q_i} dq_i .$$