Because of operators like $\vec{p}$, we generalize the notion of self-adjoint operators to hermitian operators: those for which

$$\int_a^b dx\; v^* \mathcal{L} u \;=\; \int_a^b dx\, (\mathcal{L} v)^* u$$

as long as u and v satisfy suitable boundary conditions at $x = a$ & b. We say $\mathcal{L} = \mathcal{L}^\dagger$.

Suppose $\mathcal{L} = \mathcal{L}^\dagger$, i.e., that $\mathcal{L}$ is hermitian, and that

$$\mathcal{L} u_i + \lambda_i w u_i = 0 \qquad i = 1, 2, \ldots .$$

Then also

$$\mathcal{L} u_j + \lambda_j w u_j = 0$$

so that

$$(\mathcal{L} u_j)^* + \lambda_j^* w u_j^* = 0 .$$

Note we take $w(x) = w^*(x)$ to be real. Then both

$$u_j^* \mathcal{L} u_i + \lambda_i w u_j^* u_i = 0$$

and

$$u_i (\mathcal{L} u_j)^* + \lambda_j^* w u_j^* u_i = 0$$

so that

$$\underbrace{\int_a^b dx \left[u_j^* \mathcal{L} u_i - u_i (\mathcal{L} u_j)^* \right]}_{=\,0} = (\lambda_j^* - \lambda_i) \int_a^b dx\, w(x)\, u_j^*(x)\, u_i(x) .$$

0 since $\mathcal{L} = \mathcal{L}^\dagger$, as long as the u_i's satisfy the appropriate boundary conditions.

So

$$(\lambda_j^* - \lambda_i) \int_a^b dx \; w(x) \; u_j^*(x) u_i(x) = 0 .$$

Set $i = j$. Then

$$(\lambda_i^* - \lambda_i) \int_a^b dx \; w(x) |u_i(x)|^2 = 0 .$$

Since by assumption $w(x) > 0$ except at isolated points, it follows that

$$\lambda_i^* = \lambda_i .$$

The eigenvalues of hermitian operators are <u>real</u>.

So the top equation reads

$$(\lambda_j - \lambda_i) \int_a^b dx \; w(x) \; u_j^*(x) u_i(x) = 0 .$$

Thus the eigenfunctions $u_i(x)$ and $u_j(x)$ of different, unequal eigenvalues $\lambda_j \neq \lambda_i$ must be orthogonal.

When two or more eigenfunctions do have the same eigenvalue, they are called <u>degenerate</u>. In the H atom, for instance, states with the same principal quantum number, n, are degenerate to lowest order in the non-relativistic theory.

Suppose several $u_i(x)$ all have the same λ

$$\mathcal{L} u_i + \lambda w u_i = 0 \qquad \text{for } i = 1, 2, \ldots N.$$

Then any linear combination of the u_i's also will satisfy

$$\mathcal{L}\left(\sum_{i=1}^{N} c_i u_i\right) + \lambda w \left(\sum_{i=1}^{N} c_i u_i\right) = 0$$

because $\mathcal{L}$ is a linear differential operator. So we may choose the c_i's so as to make mutually orthogonal linear combinations

$$\psi_i = \sum_{j=1}^{N} c_{ij} u_j .$$

We may even make them orthonormal

$$\phi_i = \frac{\psi_i}{\left[\int_a^b dx\, |\psi_i|^2 w\right]^{1/2}} .$$

The Gram-Schmidt way:

Set $\qquad \psi_1(x) = u_1(x)$

$$\phi_1(x) = \frac{\psi_1(x)}{\left[\int_a^b |\psi_1(x)|^2 w(x)\, dx\right]^{1/2}}$$

For $n=2$, we set

$$\psi_2(x) = u_2(x) + a_{21}\,\phi_1(x).$$

We want

$$0 = \int_a^b dx\; \psi_2(x)\,\phi_1^*(x)\,w(x)$$

$$= \int_a^b dx\; u_2(x)\,\phi_1^*(x)\,w(x) \;+\; a_{21}\int_a^b dx\,|\phi_1(x)|^2\,w(x)$$

So

$$a_{21} = -\int_a^b dx\; u_2(x)\,\phi_1^*(x)\,w(x).$$

Then

$$\phi_2(x) = \frac{\psi_2(x)}{\left[\int_a^b dx\,|\psi_2(x)|^2\,w(x)\right]^{1/2}},$$

Suppose now that $\phi_1, \phi_2, \cdots \phi_i$ are all orthonormal. We set

$$\psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^{i} a_{i+1,j}\,\phi_j(x).$$ We set

$$0 = \int_a^b dx\; \psi_{i+1}(x)\,\phi_j^*(x)\,w(x)$$

$$0 = \int_a^b dx\; u_{i+1}(x)\,\phi_j^*(x)\,w(x) \;+\; a_{i+1,j}$$ so

$$a_{i+1,j} = -\int_a^b dx\; u_{i+1}(x)\,\phi_j^*(x)\,w(x).$$

Finally

$$\phi_{i+1}(x) = \frac{\psi_{i+1}(x)}{\left[\int_a^b dx\, |\psi_{i+1}(x)|^2 w(x)\right]^{1/2}}.$$

So to gild the lilly, we write

$$\psi_{i+1}(x) = u_{i+1}(x) + \sum_{j=1}^{i} a_{i+1,j}\, \phi_j(x)$$

$$= u_{i+1}(x) - \sum_{j=1}^{i} \int_a^b dx'\, u_{i+1}(x')\, \phi_j^*(x')\, w(x')\, \phi_j(x)$$

if we set

$$P_j u_i(x) = \left[\int_a^b dt\, u_i(t)\, \phi_j^*(t) w(t)\right] \phi_j(x),$$ then

$$\psi_{i+1} = \left\{1 - \sum_{j=1}^{i} P_j\right\} u_{i+1}(x).$$

Example 9.3.1 Say $u_n(x) = x^n$ $n = 0, 1, 2, \ldots$
and the interval is $-1 \le x \le 1$ and $w(x) = 1$.
Now $u_0 = 1$, so $\psi_0 = 1$, so $\phi_0 = \frac{1}{\sqrt{2}}$.

$\psi_1 = x + a_{10}\frac{1}{\sqrt{2}}$ and so $a_{10} = -\int_{-1}^{1} \frac{x}{\sqrt{2}}\, dx = 0$

and $\phi_1(x) = \sqrt{\frac{3}{2}}\, x$.

Next

$$\psi_2 = x^2 + a_{20}\frac{1}{\sqrt{2}} + a_{21}\sqrt{\frac{3}{2}}\, x \qquad \text{and}$$

$$a_{20} = -\int_{-1}^{1} \frac{x^2 dx}{\sqrt{2}} = -\frac{\sqrt{2}}{3}$$

$$a_{21} = -\int_{-1}^{1} \sqrt{\frac{3}{2}}\, x^3 dx = 0 \qquad \text{and so}$$

$$\psi_2 = x^2 - \frac{1}{3} \quad \text{and} \quad \phi_2(x) = \sqrt{\frac{5}{2}}\,\frac{1}{2}(3x^2-1).$$

Eventually

$$\phi_3(x) = \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x).$$

It turns out that these are the Legendre polynomials

$$\phi_n(x) = \sqrt{\frac{2n+1}{2}}\; P_n(x)$$

apart from factors that reflect different normalization conditions.

Completeness

If any function $f(x)$ in a certain space of functions S can be represented as

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

in the sense that

$$0 = \lim_{N \to \infty} \int_a^b \left| f(x) - \sum_{n=0}^{N} a_n \phi_n(x) \right|^2 w(x)\,dx ,$$

then the set of functions $\phi_n(x)$ is said to span that space S or to be complete in S. The coefficients a_n are

$$\int_a^b f(x)\, \phi_n^*(x)\, w(x)\, dx = \int_a^b \sum_{j=0}^{\infty} a_j\, \phi_j(x)\, \phi_n^*(x) w(x)\, dx$$

$$= \sum_{j=0}^{\infty} a_j\, \delta_{jn} = a_n \quad \text{so}$$

$$a_n = \int_a^b f(x)\, \phi_n^*(x)\, w(x)\, dx .$$

But when $\mathcal{L} u = (p u')' + q u$ with p and q real

$$0 = \mathcal{L} u + \lambda w u \quad \text{and} \quad w \geqslant 0, \text{ then}$$

$$0 = (\mathcal{L} u)^* + \lambda w u^* = \mathcal{L} u^* + \lambda w u^* .$$

So u^* is also an eigenfunction and so we may replace u by the real function $\frac{u + u^*}{2}$ suitably normalized.

Typical spaces S are the space $\mathcal{L}_2$ of all square-integrable functions and the space $\mathcal{P}$ of all piece-wise continuous functions. The proof that the eigenfunctions of any class of hermitian operators are complete in $\mathcal{L}_2$ or $\mathcal{P}$ is beyond the scope of this course.

But in the case of the sets of orthogonal polynomials — the Legendre polynomials and others listed in Table 9.3 — we can say more. These polynomials are equivalent to the powers of x, x^n for $n \geq 0$. So we have half of a Laurent series or a whole power series.

If the ϕ_n's are complete for a space S that includes the function f, then

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \quad \text{where} \quad a_n = \int_a^b f(x) w(x) \phi_n^*(x)\, dx \quad \text{so}$$

$$f(x) = \sum_{n=0}^{\infty} \int_a^b f(y) w(y) \phi_n^*(y)\, dy \, \phi_n(x)$$

$$= \int_a^b dy\, f(y) \left[w(y) \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(y) \right] = \int_a^b dy\, f(y)\, \delta(x-y)$$

So

$$\delta(x-y) = (w(x) w(y))^{\frac{1}{2}} \sum_{n=0}^{\infty} \phi_n(x) \phi_n^*(y) .$$

Completeness leads to a formula for $\delta(x-y)$.

Bessel's inequality is

$$0 \le \int_a^b dx\, w(x)\, \left| f(x) - \sum_{i=0}^{\infty} a_i \phi_i \right|^2$$

$$0 \le \int_a^b dx\, w(x)\, |f(x)|^2 - \int_a^b dx\, w(x)\, f(x) \sum a_i^* \phi_i^*$$
$$- \int_a^b dx\, w(x)\, f^*(x) \sum a_i \phi_i + \int_a^b dx\, w \sum_{ij} a_i^* a_j \phi_i^* \phi_j \quad \text{or}$$

$$0 \le \int_a^b dx\, w(x) |f(x)|^2 - \sum |a_i|^2 - \sum |a_i|^2 + \sum |a_i|^2$$

or

$$\int_a^b dx\, w(x)\, |f(x)|^2 \ge \sum_{i=0}^{\infty} |a_i|^2 .$$

In many cases, the absolute-value signs are superfluous.

Schwarz's inequality

Let $\psi = f + \lambda g$, so that

$$0 \le \int_a^b |\psi|^2 dx = \int_a^b |f|^2 dx + \lambda \int_a^b f^* g\, dx + \lambda^* \int_a^b g^* f\, dx + |\lambda|^2 \int_a^b |g|^2 dx$$

Treating λ and λ^* as independent variables, we get

$$0 = \int_a^b f^* g\, dx + \lambda^* \int_a^b |g|^2 dx \qquad \text{and}$$

$$0 = \int_a^b g^* f\,dx + \lambda \int_a^b |g|^2\,dx$$

With these values of λ and λ^*, we get

$$0 \le \int_a^b |\psi|^2 dx = \int_a^b |f|^2 dx - \left(\int_a^b g^* f\,dx \Big/ \int_a^b |g|^2 dx\right) \int_a^b f^* g\,dx$$

$$- \int_a^b g^* f\,dx \left(\int_a^b g f^* dx \Big/ \int_a^b |g|^2 dx\right) + \frac{\int_a^b f g^* dx \int_a^b g f^* dx}{\int_a^b |g|^2 dx}$$

so that

$$\int_a^b |f|^2 dx \int_a^b |g|^2 dx \ge \left| \int_a^b g^* f\,dx \right|^2.$$

The vector analog is

$$|\vec{a}^* \cdot \vec{b}|^2 \le \vec{a}^* \cdot \vec{a}\ \ \vec{b}^* \cdot \vec{b}$$

or in Dirac notation

$$|\langle \phi | \psi \rangle|^2 \le \langle \phi | \phi \rangle \langle \psi | \psi \rangle.$$

So when ϕ & ψ are normalized, the probability of finding ψ as ϕ is ≤ 1

$$P(\phi, \psi) \le 1.$$

The review on pages 609–613 is worth reading.

Suppose $\mathcal{L} = \mathcal{L}^\dagger$ is a hermitian operator with eigenfunctions ϕ_n and eigenvalues λ_n

$$\mathcal{L}\phi_n + \lambda_n \phi_n = 0.$$

So $w(x) = 1$ here. Build the Green's function

$$G(x,y) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}.$$

See

$$(\mathcal{L} + \lambda) G(x,y) = \sum_{n=0}^{\infty} \frac{(\mathcal{L}+\lambda)\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}$$

$$= \sum_{n=0}^{\infty} (\lambda - \lambda_n) \frac{\phi_n(x)\phi_n^*(y)}{\lambda_n - \lambda}$$

$$= - \sum_{n=0}^{\infty} \phi_n(x)\phi_n^*(y) = -\delta(x-y).$$

So

$$(\mathcal{L} + \lambda) G(x,y) = -\delta(x-y).$$

So if we have $\mathcal{L}\psi + \lambda\psi = -\rho$, then we try

$$\psi(x) = \int d^3y \, G(x,y)\rho(y) \quad \text{so that}$$

$$(\mathcal{L}+\lambda)\psi(x) = \int d^3x \, (\mathcal{L}+\lambda) G(x,y)\rho(y) = -\int d^3y \, \delta(x-y)\rho(y) = -\rho(x).$$

A more explicit treatment is available in one dimension: We will take $\mathcal{L}$ to be real and self adjoint

$$\mathcal{L}u = (pu')' + qu$$ and we want to solve

$$\mathcal{L}y(x) + f(x) = 0.$$

Try $$G(x,y) = -\frac{1}{A}\begin{cases} u(x)v(y) & a \le x < y \\ u(y)v(x) & y < x \le b \end{cases}$$

here A is a constant

and $\mathcal{L}u = \mathcal{L}v = 0$ and u and v respectively satisfy suitable boundary conditions at $x = a$ and $x = b$:

$$u(a)\,u'(a) = 0 \quad \text{or} \quad \alpha u(a) + \beta u'(a) = 0$$
$$v(b)\,v'(b) = 0 \quad \text{or} \quad \alpha v(b) + \beta v'(b) = 0.$$

Set
$$y(x) = \int_a^b dy\, G(x,y)\, f(y)$$

$$= -\frac{1}{A}\int_a^x dy\, v(x)u(y)f(y) - \frac{1}{A}\int_x^b dy\, u(x)v(y)f(y)$$

So

$$y'(x) = -\frac{v'(x)}{A}\int_a^x dy\, u(y)f(y) - \frac{u'(x)}{A}\int_x^b dy\, v(y)f(y)$$

$$\left(-\frac{1}{A}v(x)u(x)f(x) + \frac{1}{A}u(x)v(x)f(x) = 0\right)$$

$$y''(x) = -\frac{v''(x)}{A}\int_a^x dy\, u(y)\, f(y) \;-\; \frac{u''(x)}{A}\int_x^b dy\, v(y)\, f(y)$$

$$-\frac{1}{A}\left[u(x)\,v'(x) - u'(x)\,v(x)\right] f(x).$$

Wronsky strikes again! Note that since u & v satisfy

$$0 = \mathcal{L}u = (pu')' + qu = \mathcal{L}v = (pv')' + qv = 0,$$

the Wronskian

$$W = uv' - u'v \qquad \text{satisfies}$$

$$W' = uv'' - u''v$$

Now
$0 = \mathcal{L}u$ implies $pu'' = -p'u' - qu$ so $u'' = -\frac{p'}{p}u' - \frac{q}{p}u$
and $0 = \mathcal{L}v$ implies $v'' = -\frac{p'}{p}v' - \frac{q}{p}v$. So

$$W' = u\left(-\frac{p'v'}{p} - \frac{qv}{p}\right) - \left(-\frac{p'u'}{p} - \frac{q}{p}u\right)v$$

$$= -\frac{p'}{p}(uv' - vu') = -\frac{p'}{p}W \qquad \text{So}$$

$$\frac{W'}{W} = -\frac{p'}{p} \qquad (\log W)' = -(\log p)'$$

So

$$\log W = -\log p + c$$

$$W = \frac{A}{p}. \qquad \text{So} \qquad \frac{W}{A} = \frac{1}{p}.$$

So

$$y''(x) = -\frac{v''(x)}{A}\int_a^x u(y)f(y)dy - \frac{u''(x)}{A}\int_x^b v(y)f(y)dy - \frac{f(x)}{p(x)}$$

So

$$py'' + p'y' = \mathcal{L}y = \frac{-pv'' - p'v'}{A}\int_a^x u(y)f(y)dy - \frac{pu'' - p'u'}{A}\int_x^b v(y)f(y)dy$$

$$- f(x) \qquad \text{or}$$

$$\mathcal{L}y = -\frac{\mathcal{L}v}{A}\int_a^x u(y)f(y)dy - \frac{\mathcal{L}u}{A}\int_x^b v(y)f(y)dy - f(x)$$

But $\mathcal{L}u = \mathcal{L}v = 0$. So

$$\mathcal{L}y + f = 0 \quad \text{or} \quad \mathcal{L}y(x) + f(x) = 0.$$

Note that $y(x)$ satisfies the same boundary conditions at $x = a, b$ as u and v:

$$y(a) = -\frac{1}{A}\int_a^b dy\, u(a)v(y)f(y) = \left(-\frac{1}{A}\int_a^b dy\, v(y)f(y)\right) u(a)$$

$$y'(a) = \left(-\frac{1}{A}\int_a^b dy\, v(y)f(y)\right) u'(a)$$

$$y(b) = \left(-\frac{1}{A}\int_a^b dy\, u(y)f(y)\right) v(b)$$

$$y'(b) = \left(-\frac{1}{A}\int_a^b dy\, u(y)f(y)\right) v'(b).$$

The function $\frac{1}{|\vec{R}-\vec{r}|}$ occurs throughout electrodynamics and gravity theory. If $R > r$, then we may expand it as a power series in r/R:

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{(R^2+r^2-2\vec{R}\cdot\vec{r})^{\frac{1}{2}}} = \frac{1}{(R^2+r^2-2Rr\cos\theta)^{1/2}}.$$

Letting $x = \cos\theta$ and $t = r/R$, we have

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R(1+t^2-2xt)^{1/2}} = \frac{1}{R}\, g(t,x)$$

where $g(t,x)$ may be expanded in a power series in t for $t<1$

$$g(t,x) = (1+t^2-2xt)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n .$$

Since $t = r/R$, the Green's function is

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{R}\right)^n , \qquad \left(\vec{r}\cdot\vec{R} = rR\cos\theta\right)$$

which is often written as

$$\frac{1}{|r-r'|} = \frac{1}{r_>}\sum_{n=0}^{\infty}\left(\frac{r_<}{r_>}\right)^n P_n(\cos\theta).$$

The function $g(t,x)$ is a generating function for the Legendre polynomials $P_n(x)$.

$$\sum_{n=0}^{\infty} t^n P_n(x) = g(t,x) = (1+t^2-2tx)^{-1/2}$$

Using the binomial theorem of Sec. 5.6, Eq. (5.103), we have

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m!}{n!\,(m-n)!} x^n = \sum_{n=0}^{\infty} \binom{m}{n} x^n,$$

which converges for $|x|<1$. If m is an integer, then the series terminates at $n=m$ since $(m-n)! = \pm\infty$ for $n>m$. If m is not an integer, then:

$$\binom{m}{n} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}.$$

After some algebra, Ex. (10.1.15), one has

$$(1+t^2-2tx)^{-\frac{1}{2}} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} (2xt-t^2)^n$$

where $n!! = n(n-2)(n-4)\cdots$ down to 2 or 1. One has

$P_0(x)=1$, $\quad P_1(x)=x$, $\quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, and

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!} x^{n-2k},$$

where $[5/2] = [2.5] = 2$, etc., $[x]$ is the greatest integer not greater than x.

Now

$$g(t,x) = (1 + t^2 - 2tx)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n, \quad \text{so}$$

$$\frac{\partial g}{\partial t}(t,x) = \frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} n\, P_n(x)\, t^{n-1}$$

So

$$(x-t)\, g(t,x) = (1-2xt+t^2) \sum_{n=0}^{\infty} n\, P_n(x)\, t^{n-1}$$

$$= (x-t) \sum_{n=0}^{\infty} P_n(x)\, t^n$$

By identifying the coefficients of t^n, one has

$$(2n+1)\, x\, P_n(x) = (n+1)\, P_{n+1}(x) + n\, P_{n-1}(x) \qquad (*)$$

e.g.

$$3x P_1 = 2P_2 + P_0 \quad \text{so}$$

$$P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

Computationally one uses the recurrence relation

$$P_{n+1}(x) = 2x\, P_n(x) - P_{n-1}(x) - [x P_n(x) - P_{n-1}(x)]/(n+1).$$

It's more stable than the three-term recurrence relation $(*)$.

$$\frac{\partial g(t,x)}{\partial x} = \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)\, t^n$$

or

$$(1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n - t\sum_{n=0}^{\infty} P_n(x)\, t^n = 0, \text{ so}$$

setting to zero the coefficient of t^n, we get

$$P_{n+1}'(x) + P_{n-1}'(x) = 2x\, P_n'(x) + P_n(x). \qquad (**)$$

We differentiate Eq. (*) and combine it with (**) to get

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)\, P_n(x), \qquad (+)$$

These last two relations lead to yet more:

$$P_{n+1}'(x) = (n+1) P_n(x) + x\, P_n'(x)$$

$$P_{n-1}'(x) = -n\, P_n(x) + x\, P_n'(x)$$

$$(1-x^2)\, P_n'(x) = n\, P_{n-1}(x) - n x\, P_n(x)$$

$$(1-x^2)\, P_n'(x) = (n+1)\, x\, P_n(x) - (n+1)\, P_{n+1}(x)$$

and

$$(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0 \qquad (!)$$

This last equation is Legendre's equation. It is self adjoint:

$$\left[(1-x^2)P_n'(x)\right]' + n(n+1)\,P_n(x) = 0.$$

With $x = \cos\theta$ $\qquad 1-x^2 = \sin^2\theta$

$$\frac{d}{d\theta} = \frac{d\cos\theta}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx} \qquad \text{so}$$

$$\frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

and L's eq is

$$-\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin^2\theta\left(-\frac{1}{\sin\theta}\right)\frac{d}{d\theta}P_n(\cos\theta)\right] + n(n+1)P_n(\cos\theta) = 0$$

or

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}P_n(\cos\theta)\right] + n(n+1)P_n(\cos\theta) = 0$$

or

$$\frac{d^2}{d\theta^2}P_n(\cos\theta) + \cot\theta\frac{d}{d\theta}P_n(\cos\theta) + n(n+1)P_n(\cos\theta) = 0.$$

$$g(t,1) = \frac{1}{(1-2t+t^2)^{\frac{1}{2}}} = \frac{1}{((1-t)^2)^{1/2}} = \frac{1}{1-t}$$

$$= \sum_{n=0}^{\infty} t^n = \sum_{n=0}^{\infty} P_n(1)\, t^n$$

So

$$P_n(1) = 1.$$

$$g(t,-1) = \frac{1}{1+t} = \sum_{n=0}^{\infty} (-t)^n = \sum_{n=0}^{\infty} P_n(-1)\, t^n,$$

so

$$P_n(-1) = (-1)^n.$$

Similarly but with much more effort one shows that

$$P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!}$$

and

$$P_{2n+1}(0) = 0, \qquad \text{(zip)}$$

Parity

$(1+t^2-2tx)^{-\frac{1}{2}} = g(t,x) = g(-t,-x)$ so

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(-x)\,(-t)^n = g(-t,-x).$$

So $\quad P_n(-x) = (-1)^n P_n(x)$ which implies (zip).

Eqs. (12.38 – 12.39c) of the text show that

$$P_n(\cos\theta) = \sum_{m=0 \text{ or } 1}^{n} a_m \cos m\theta \qquad \text{all } a_m \geq 0$$

So the maximum of $P_n(\cos\theta)$ is at $\theta = 0$

$$|P_n(x)| = |P_n(\cos\theta)| \leq P_n(1) = 1.$$

Orthogonality

$$[(1-x^2)P_n'(x)]' + n(n+1)P_n(x) = 0$$

is a self-adjoint ODE. So its eigenfunctions associated with different eigenvalues are orthogonal. Integrating by parts and dropping boundary terms which vanish with $(1-x^2)$ at $x=\pm 1$, we get

$$\int_{-1}^{1} dx \left\{ P_m(x)[(1-x^2)P_n'(x)]' - P_n[(1-x^2)P_m'(x)]' \right\}$$

$$= \int_{-1}^{1} dx \left\{ -P_m'(x)[(1-x^2)P_n'(x)] + P_n'(x)[(1-x^2)P_m'(x)] \right\}$$

$$= 0 = [m(m+1) - n(n+1)] \int_{-1}^{1} P_n(x)P_m(x)\,dx$$

So if $m \neq n$

$$\int_{-1}^{1} dx\, P_n(x) P_m(x) = 0 \qquad \text{or}$$

$$\int_{0}^{\pi} P_n(\cos\theta)\, P_m(\cos\theta) \sin\theta\, d\theta = 0$$

After some algebra (Eqs. (12.44) — (12.48) of text), one has

$$\int_{-1}^{1}\left[P_n(x)\right]^2 dx = \frac{2}{2n+1} .$$

So

$$\int_{-1}^{1} dx \quad P_n(x) \; P_m(x) = \frac{2}{n+1}\,\delta_{nm} .$$

Suppose we expand on $[-1, 1]$

$$f(x) = \sum_{m=0}^{\infty} a_m \, P_m(x) ,$$

Then

$$\int_{-1}^{1} dx \;\; P_n(x)\, f(x) = \sum_{m=0}^{\infty} a_m \int_{-1}^{1} dx\, P_n(x) P_m(x)$$

$$= \sum_{m=0}^{\infty} a_m \frac{2}{n+1}\,\delta_{nm}$$

$$= \frac{2\,a_n}{n+1} , \text{ or}$$

$$a_n = \frac{n+1}{2} \int_{-1}^{1} dx \;\, P_n(x)\, f(x) , \qquad \text{and so}$$

$$f(x) = \sum_{n=0}^{\infty} a_n \, P_n(x) = \sum_{n=0}^{\infty} \frac{n+1}{2}\, P_n(x) \int_{-1}^{1} dy \; P_n(y)\, f(y) .$$

But this means that

$$f(x) = \int_{-1}^{1} dy\, f(y) \left[\sum_{n=0}^{\infty} \frac{n+1}{2} P_n(x) P_n(y) \right]$$

$$= \int_{-1}^{1} dy\, f(y) \quad \delta(x-y)$$

So on $[-1,1]$ and for the space of functions spanned by the $P_n(x)$, the Dirac delta function is

$$\delta(x-y) = \sum_{n=0}^{\infty} \frac{n+1}{2} P_n(x) P_n(y) .$$

Rodrigues's formula is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n .$$

Schlaefli's integral.

$$(x^2-1)^n = \frac{1}{2\pi i} \oint \frac{(z^2-1)^n}{z-x} dz$$

$$\frac{1}{2^n n!} \frac{d\,(x^2-1)^n}{dx^n} = P_n(x) = \frac{1}{2^n 2\pi i} \oint \frac{(z^2-1)^n}{(z-x)^{n+1}} dz .$$