The inner product of $|\vec{p}\rangle$ with $|\vec{x}\rangle$ is

$$\langle\vec{x}'|\vec{p}'\rangle = \frac{e^{i\vec{x}'\cdot\vec{p}'/\hbar}}{(2\pi\hbar)^{3/2}},$$

in which $\vec{x}'$ and $\vec{p}'$ are the eigenvalues (ev) of the operators $\vec{x}$ and $\vec{p}$

$$\vec{x}\,|\vec{x}'\rangle = \vec{x}'\,|x'\rangle$$

$$\vec{p}\,|\vec{p}'\rangle = \vec{p}'\,|p'\rangle.$$

These eigenvectors $|\vec{x}'\rangle$ and $|\vec{p}'\rangle$ are complete

$$1 = \int d^3x'\, |\vec{x}'\rangle\langle\vec{x}'| = \int d^3p'\, |\vec{p}'\rangle\langle\vec{p}'|$$

so that

$$1 = \int d^3x'\, |\vec{x}'\rangle\langle\vec{x}'| \int d^3p'\, |\vec{p}'\rangle\langle\vec{p}'| \int d^3x''\, |\vec{x}''\rangle\langle\vec{x}''|$$

$$= \int d^3x'\, d^3p'\, d^3x''\, |\vec{x}'\rangle\langle\vec{x}''|\, \frac{e^{i(\vec{x}'-\vec{x}'')\cdot\vec{p}'/\hbar}}{(2\pi\hbar)^3}$$

Let $\vec{k} = p'/\hbar$

$$1 = \int d^3x'\,d^3x''\, d^3k\, \frac{e^{i(\overrightarrow{x'-x''})\cdot\vec{k}}}{(2\pi)^3}\, |\vec{x}'\rangle\langle\vec{x}''| = \int d^3x'\,d^3x''\,|\vec{x}'\rangle\langle x''|\,\delta^{(3)}(\overrightarrow{x'-x''})$$

$$= \int d^3x'\, |\vec{x}'\rangle\langle\vec{x}'| = 1.$$

Consider the gaussian wave packet

$$\langle \vec{x}' | \psi \rangle = \psi(\vec{x}') = \left(\frac{1}{\pi^{1/4} \sqrt{d}} \right)^3 e^{i \vec{k} \cdot \vec{x}' - \frac{\vec{x}'^2}{2d^2}}$$

which is a plane wave of wave number $\vec{k} = \vec{p}/\hbar$ modulated by a gaussian profile. The probability density

$$P(\vec{x}') = |\langle \vec{x}' | \psi \rangle|^2$$

$$= \left(\frac{1}{d\sqrt{\pi}} \right)^3 e^{-\frac{\vec{x}'^2}{d^2}}$$

drops sharply away from the origin $\vec{x}' = \vec{0}$. So we have a particle of momentum nearly $\hbar \vec{k}$ nearly at $\vec{x}' = \vec{0}$.

The mean value of $\vec{x}'$ vanishes

$$\langle \vec{x} \rangle = \int d^3x' \; P(\vec{x}') \vec{x}' = \int d^3x' \frac{\vec{x}'}{(d\sqrt{\pi})^3} e^{-\vec{x}'^2/d^2} = 0,$$

but

$$\langle \vec{x}^2 \rangle = \int d^3x' \; P(\vec{x}') \vec{x}'^2$$

$$= \frac{1}{(d\sqrt{\pi})^3} \int d^3x' \; \vec{x}'^2 e^{-\vec{x}'^2/d^2}$$

$$= \frac{1}{(d\sqrt{\pi})^3}\left(-\frac{d}{d\,d^{-2}}\right)\int d^3x'\, e^{-x'^2/d^2}$$

$$= \frac{1}{(d\sqrt{\pi})^3}\left(\frac{d}{d(-d^{-2})}\right) d^3\int d^3y\, e^{-\vec{y}^{\,2}}$$

let $\alpha = \frac{1}{d^2}$

$$= \frac{\int d^3y\, e^{-\vec{y}^{\,2}}}{(d\sqrt{\pi})^3}\left(-\frac{d}{d\alpha}\,\alpha^{-3/2}\right)$$

$$= \frac{3}{2}\,\alpha^{-5/2}\int\frac{d^3y\, e^{-\vec{y}^{\,2}}}{(d\sqrt{\pi})^3}$$

$$= \frac{3}{2}\,\frac{d^5}{d^3}\int\frac{d^3y}{(\sqrt{\pi})^3}\, e^{-\vec{y}^{\,2}}$$

Recall

$$\left(\int_{-\infty}^{\infty}dx\, e^{-x^2}\right)^2 = \int dx\,dy\, e^{-x^2-y^2} = \int 2\pi r\,dr\, e^{-r^2} = \pi$$

So

$$\langle\vec{x}'^2\rangle = \frac{3}{2}d^2 \quad \text{and also}$$

$$\int d^3x'\, P(x') = \int\frac{d^3x'}{(d\sqrt{\pi})^3}\, e^{-\frac{\vec{x}^2}{d^2}} = \int\frac{d^3y}{\pi^{3/2}}\, e^{-y^2} = 1.$$

So $P(x')$ is normalized.

What is $\langle \vec{p} \rangle$? As HW 4, problem 1, show

$$\langle \vec{p} \rangle = \int d^3\vec{x}' \, \psi^*(\vec{x}') \, \frac{\hbar}{i} \nabla \psi(\vec{x}') = \hbar k = \vec{p}$$

and

$$\langle \vec{p}^{\,2} \rangle = \int d^3x' \, \psi^*(\vec{x}') \left[-\hbar^2 \nabla^2 \psi(\vec{x}') \right].$$

$$= \frac{3\hbar^2}{2d^2} + \hbar^2 \vec{k}^2 = \frac{3\hbar^2}{2d^2} + \vec{p}'^{\,2}.$$

It follows then that

$$\langle (\Delta x)^2 \rangle = \langle (\vec{x} - \langle \vec{x}' \rangle)^2 \rangle$$

$$= \langle \vec{x}^{\,2} - 2\langle \vec{x} \rangle^2 + \langle \vec{x} \rangle^2 \rangle$$

$$= \langle \vec{x}^{\,2} \rangle - \langle \vec{x} \rangle^2 = \langle \vec{x}^{\,2} \rangle = \frac{3}{2} d^2$$

and that

$$\langle (\Delta \vec{p})^2 \rangle = \langle (\vec{p}' - \langle \vec{p} \rangle)^2 \rangle$$

$$= \langle \vec{p}^{\,2} \rangle - \langle \vec{p} \rangle^2$$

$$= \frac{3}{2} \frac{\hbar^2}{d^2} + \hbar^2 \vec{k}^2 - \hbar^2 \vec{k}^2$$

$$= \frac{3}{2} \frac{\hbar^2}{d^2}.$$

Let's look at a single degree of freedom, say x_1 and p_1. Then

$$\langle(\Delta x_1)^2\rangle = \frac{1}{2} d^2$$

and

$$\langle(\Delta p_1)^2\rangle = \frac{1}{2}\frac{\hbar^2}{d^2},$$

so the product is

$$\langle(\Delta x_1)^2\rangle\langle(\Delta p_1)^2\rangle = \frac{1}{4}\hbar^2 \qquad \text{or}$$

$$\Delta x_1 \, \Delta p_1 = \frac{\hbar}{2}.$$

The momentum-space amplitude is

$$\langle \vec{p}\,'|\psi\rangle = \int d^3x' \, \langle \vec{p}\,'|\vec{x}\,'\rangle\langle x'|\psi\rangle$$

$$= \int \frac{d^3x' \, e^{-i p'\cdot x'/\hbar}}{(2\pi\hbar)^{3/2}} \frac{1}{(\pi^{1/4}\sqrt{d})^3} e^{ik\cdot x' - x'^2/2d^2}$$

$$= \frac{e^{-\frac{d^2}{2\hbar^2}(p'-\hbar k)^2}}{(2\pi\hbar)^{3/2}(\pi^{1/4}\sqrt{d})^3}\int d^3x' \, e^{-\frac{(\vec{x}\,' + i d^2(\vec{p}\,'/\hbar - \vec{k}))^2}{2d^2}}$$

Now the single integral

$$\int_{-\infty}^{\infty} dx \; e^{-\frac{(x+i\alpha)^2}{2d^2}} = \int_{-\infty}^{\infty} dx \; e^{-\frac{x^2}{2d^2}} = d\sqrt{2}\int_{-\infty}^{\infty} dy\, e^{-y^2} = d\sqrt{2\pi}$$

So

$$\langle \vec{p}\,' | \psi \rangle = \frac{d^{3/2}}{\hbar^{3/2}} \frac{1}{\pi^{3/4}} e^{-\frac{d^2}{2\hbar^2}(\vec{p}\,' - \hbar \vec{k})^2}$$

What is $\psi(\vec{x}, t)$? We take

$$H = \frac{\vec{p}^{\,2}}{2m} \quad \text{where} \quad \vec{p}\,|\vec{p}\,'\rangle = \vec{p}\,'\,|\vec{p}\,'\rangle.$$

$$\langle \vec{x}\,' | \psi, t \rangle = \langle \vec{x} | e^{-iHt/\hbar} | \psi \rangle$$

$$= \langle \vec{x} | e^{-i\frac{Ht}{\hbar}} \int d^3p' \, |\vec{p}\,'\rangle\langle \vec{p}\,' | \psi \rangle$$

$$= \int d^3p' \, \langle \vec{x} | p' \rangle \, e^{-i\frac{p'^2 t}{2m\hbar}} \langle \vec{p}\,' | \psi \rangle$$

$$= \int d^3p' \, \frac{e^{i\vec{x}\cdot\vec{p}\,'/\hbar}}{(2\pi\hbar)^{3/2}} \, e^{-i\frac{p'^2 t}{2m\hbar}} \, \frac{d^{3/2}}{\hbar^{3/2}} \frac{1}{\pi^{3/4}} e^{-\frac{d^2}{2\hbar^2}(\vec{p}\,' - \hbar\vec{k})^2}$$

HW 4, Problem 2, Find $\langle \vec{x}\,' | \psi, t \rangle$.

The Laplace transform $f(s)$ of a function $F(t)$ is

$$f(s) = \int_0^\infty dt \; e^{-st} \; F(t) = \mathcal{L}\{F(t)\}.$$

So if $F(t) = 1$, then

$$f(s) = \int_0^\infty dt \; e^{-st} = \left.\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s} .$$

If $F(t) = e^{kt}$, then

$$f(s) = \int_0^\infty dt \; e^{-st} e^{kt} = \frac{1}{s-k} \quad \text{for } s > k.$$

So if $F(t) = \operatorname{ch} kt$, then

$$f(s) = \int_0^\infty dt \; e^{-st} \left(\frac{e^{kt} + e^{-kt}}{2} \right)$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k} \right) = \frac{s}{s^2 - k^2} .$$

And if $F(t) = \operatorname{sh} kt$, then

$$f(s) = \int_0^\infty dt \; e^{-st} \left(\frac{e^{kt} - e^{-kt}}{2} \right) = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k} \right) = \frac{k}{s^2 - k^2} .$$

Similarly if $F(t) = \cos kt$

$$f(s) = \int_0^\infty dt\, e^{-st} \frac{e^{ikt} + e^{-ikt}}{2} = \frac{1}{2}\left(\frac{1}{s-ik} + \frac{1}{s+ik}\right)$$

$$= \frac{s}{s^2+k^2}$$ and

if $F(t) = \sin kt$, then

$$f(s) = \int_0^\infty dt\, e^{-st} \left(\frac{e^{ikt} - e^{-ikt}}{2i}\right) = \frac{1}{2i}\left(\frac{1}{s-ik} - \frac{1}{s+ik}\right)$$

$$= \frac{k}{s^2+k^2} .$$

Since

$$\frac{1}{s} = \int_0^\infty dt\, e^{-st}$$

$$-\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} = \int_0^\infty dt\, t\, e^{-st}$$

$$\left(-\frac{d}{ds}\right)^2 \frac{1}{s} = \frac{2}{s^3} = \int_0^\infty dt\, t^2 e^{-st}$$

$$\left(-\frac{d}{ds}\right)^n \frac{1}{s} = \frac{n!}{s^{n+1}} = \int_0^\infty dt\, t^n e^{-st}, \quad \text{for } s>0.$$

If $f(s) = \int_0^\infty dt\, e^{-st} F(t) = \mathcal{L}\{F(t)\}$, then

$$\mathcal{L}\{F'(t)\} = \int_0^\infty dt\, e^{-st} \frac{d}{dt} F(t)$$

$$= \int_0^\infty dt \frac{d}{dt}\left[e^{-st} F(t)\right] - F(t) \frac{d}{dt} e^{-st}$$

$$= \int_0^\infty dt\, s\, e^{-st} F(t) - F(0)$$

$$= s\,\mathcal{L}\{F(t)\} - F(0).$$

So

$$\mathcal{L}\{F''(t)\} = s\,\mathcal{L}\{F'(t)\} - F'(0)$$

$$= s\left[s\,\mathcal{L}\{F(t)\} - F(0)\right] - F'(0)$$

$$= s^2 \mathcal{L}\{F(t)\} - sF(0) - F'(0).$$

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty dt\, e^{-st}\, \delta(t-t_0) = e^{-st_0} \qquad t_0 \geq 0$$

$$\mathcal{L}\{\delta(t)\} = 1.$$

How does one invert

$$f(s) = \int_0^\infty dt \; e^{-st} F(t) \quad ?$$

Well, consider the integral

$$\int_{-\infty}^{\infty} f(is) \frac{ds}{2\pi} e^{ist} = \int_{-\infty}^{\infty} ds \int_0^\infty \frac{dt'}{2\pi} e^{is(t-t')} F(t') = F(t)$$

So the inverse is

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{ist} f(is) .$$

But a Laplace transform $f(s)$ tends to be smoother as Re s increases. So one may need to use

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{(x+is)t} f(x+is) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} \int_0^\infty dt' \, e^{(x+is)t} e^{-(x+is)t'} F(t')$$

$$= \int_0^\infty dt' \, e^{x(t-t')} \delta(t-t') F(t') = F(t).$$

The inversion formula

$$F(t) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} e^{(x+is)t} f(x+is)$$

is called a Bromwich integral or a Fourier-Mellin integral.

Homogeneous partial differential equations often possess very simple solutions. For example, the wave equation

$$(\Box - m^2c^2)\,\phi(x,t) = 0 = \left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - m^2c^2\right)\phi(x,t) = 0$$

may be solved by writing $\phi(x,t) = F(\vec{p}\cdot\vec{x} - Et)$ where F is any harmonic function $F'' = -F$ and $E^2 = \vec{p}^{\,2}c^2 + m^2c^4$.

$$\vec{\nabla}\phi(x,t) = F'\,\vec{\nabla}(p\cdot x - Et) = \vec{p}\,F'$$

and so

$$\Delta\phi = \vec{p}^{\,2}F'' = -\vec{p}^{\,2}F$$

Also

$$\frac{1}{c}\frac{\partial}{\partial t}\phi = -\frac{E}{c}F' \qquad \text{and so}$$

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = +\frac{E^2}{c^2}F'' = -\frac{E^2}{c^2}F$$

So

$$\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - m^2\right)\phi(x,t) = -\vec{p}^{\,2}F + \frac{E^2}{c^2}F - m^2c^2F$$

$$= \left(\frac{E^2}{c^2} - p^2 - m^2c^2\right)F = 0$$

as long as

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4.$$