Gibbs's Overshoot: when a Fourier series is used to represent a discontinuous function, the constraint of periodicity, which arises from the trig functions, leads to errors. For instance, the square-wave

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

has a Fourier series expansion that satisfies

$$f(0) = 0 \quad , \quad \text{which is okay,}$$

but also rises to 1.18 for some $0 < x \ll \pi$ and sinks to -1.18 for some $-\pi \ll x < 0$, as illustrated in Fig. 14.11 of page 896 of A&W.

The rest of chapter 14 is worth reading, but in class we shall go on to a discussion of the Fourier transform.

The Fourier Transform.

If $f(x)$ is piecewise continuous, piecewise differentiable, and absolutely integrable —

$$\infty > \int_{-\infty}^{\infty} |f(x)|\, dx \quad -$$

then it has a Fourier transform representation:

$$f(x) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{-i\omega x}\, g(\omega)$$

where

$$g(\omega) = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t}\, f(t).$$

The natural space of the Fourier transform is the space of square-integrable functions (L_2), those whose integrals

$$\infty > \int_{-\infty}^{\infty} |f(x)|^2\, dx \qquad \text{converge.}$$

In fact, if $f(x)$ is L_2, then so is $g(\omega)$ and

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |g(\omega)|^2\, d\omega .$$

We may derive these relations from the exponential Fourier series for $f(x)$:

$$\tilde{f}(\theta) = \sum_{n=-\infty}^{\infty} e^{-in\theta} c_n$$

where

$$c_n = \int_{-\pi}^{\pi} \tilde{f}(\theta)\, e^{+in\theta} \frac{d\theta}{2\pi},$$

which we obtained from Laurent's expansion of an analytic function, $F(z) = F(e^{i\theta}) = f(\theta)$.

First, we let $\theta = \pi x / L$

so that

$$C_n = \int_{-L}^{L} \frac{\pi}{L} \frac{1}{2\pi} \hat{f}(\pi x/L)\, e^{+in\pi x/L}\, dx$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} \hat{f}(\pi x/L)\, e^{+in\pi x/L}\, dx .$$

Now we set $f(x) = \tilde{f}(\pi x/L)$ so that

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)\, e^{+in\pi x/L}\, dx .$$

Now we set $\omega = n\pi/L$ so that

$$c_\omega = c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{i\omega x}\, dx$$

Now put $c_\omega = g(\omega)\dfrac{\sqrt{2\pi}}{2L}$ so that

$$g(\omega) = \int_{-L}^{L} f(x)\, e^{i\omega x}\, \frac{dx}{\sqrt{2\pi}} .$$

Then the expansion is

$$\tilde{f}(\theta) = f(x) = \sum_{n=-\infty}^{\infty} e^{-in\theta} c_n$$

$$= \sum_{n=-\infty}^{\infty} e^{-in\pi x/L} c_\omega$$

$$= \sum_{n=-\infty}^{\infty} e^{-i\omega x} g(\omega)\frac{\sqrt{2\pi}}{2L}$$

Now put $d\omega = \dfrac{2\pi}{2L}$ so that

$$f(x) = \int_{-\infty}^{\infty} e^{-i\omega x} g(\omega)\frac{d\omega}{\sqrt{2\pi}} \quad \text{and}$$

$$g(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\frac{dx}{\sqrt{2\pi}} .$$

The Fourier series relations

$$\tilde{f}(\theta) = \sum_{n=-\infty}^{\infty} e^{-in\theta} c_n = \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\, \tilde{f}(\phi)\, e^{in(\phi-\theta)}$$

$$= \int_{-\pi}^{\pi} d\phi\, \tilde{f}(\phi) \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\theta)}}{2\pi}$$

tell us that the sum is a Dirac delta function

$$\delta(\phi-\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\phi-\theta)} .$$

Similarly, the Fourier transform relations

$$f(x) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{-i\omega x}\, g(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, e^{-i\omega x} \int_{-\infty}^{\infty} e^{i\omega y} f(y) \frac{dy}{\sqrt{2\pi}}$$

$$= \int_{-\infty}^{\infty} dy\, f(y)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega(y-x)}$$

tell us that the inner integral is a delta function

$$\delta(y-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega(y-x)} .$$

Let's now write the integral

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} dx \, f^*(x) \, f(x)$$

in terms of f's Fourier transforms:

$$f(x) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega x} g(\omega)$$

$$f^*(x) = \int_{-\infty}^{\infty} \frac{d\psi}{\sqrt{2\pi}} e^{+i\psi x} g^*(\psi)$$

obtaining

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{d\psi}{\sqrt{2\pi}} e^{i\psi x} g^*(\psi) \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega x} g(\omega)$$

But

$$\delta(\psi - \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \; e^{ix(\psi - \omega)} \quad , \quad \text{so}$$

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} d\psi \int_{-\infty}^{\infty} d\omega \; g^*(\psi) \, g(\omega) \, \delta(\psi - \omega) = \int_{-\infty}^{\infty} d\omega \, |g(\omega)|^2 ,$$

which is called Parseval's relation.

The Convolution of two functions

$f(x)$ and $g(x)$ is defined as

$$f*g\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, f(x-y)\, dy\,.$$

The Fourier transform (FT) of $f*g(x)$ is

$$\widetilde{f*g}\,(\omega) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_{-\infty}^{\infty} dy\, e^{i\omega x}\, g(y)\, f(x-y)$$

But

$$g(y) = \int_{-\infty}^{\infty} \frac{d\psi}{\sqrt{2\pi}}\, \tilde{g}(\psi)\, e^{-i\psi y}$$

and

$$f(x-y) = \int_{-\infty}^{\infty} \frac{d\theta}{\sqrt{2\pi}}\, \hat{f}(\theta)\, e^{-i\theta(x-y)}\,, \text{ so now}$$

$$\widetilde{f*g}\,(\omega) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{2\pi}\, e^{i\omega x}\, \tilde{g}(\psi)\, e^{-i\psi y}\, \tilde{f}(\theta)\, e^{-i\theta(x-y)}\, d\psi\, d\theta$$

$$= \int_{-\infty}^{\infty} \delta(\theta-\psi)\,\delta(\omega-\theta)\, \tilde{g}(\psi)\, \hat{f}(\theta)\, d\psi\, d\theta$$

$$= \int_{-\infty}^{\infty} d\theta\, \delta(\omega-\theta)\, \tilde{g}(\theta)\, \hat{f}(\theta) = \tilde{g}(\omega)\, \tilde{f}(\omega)\,.$$