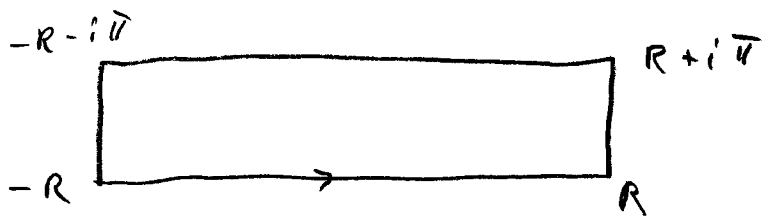
Problem 7.2.17(b)

$$I = \int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx$$

We let $x = e^y, \quad dx = e^y\,dy$

$$I = \int_{-\infty}^{\infty} \frac{y^2 e^y\,dy}{1+e^{2y}} = \int_{-\infty}^{\infty} \frac{y^2\,dy}{e^y + e^{-y}}$$



Now $$\oint_{\square} \frac{z^2\,dz}{e^z + e^{-z}} = I + \int_{R+i\pi}^{-R+i\pi} \frac{z^2\,dz}{e^z + e^{-z}}$$

$$= I + \int_R^{-R} \frac{(z+i\pi)^2\,dz}{-e^z - e^{-z}} = I + \int_{-R}^{R} \frac{(z+i\pi)^2}{e^z + e^{-z}}\,dz$$

$$= I + I + 2\pi i \int_{-R}^{R} \frac{z\,dz}{e^z + e^{-z}} - \pi^2 \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} \qquad (*)$$

The third term vanishes by symmetry, $\int_{-R}^{R} \frac{z\,dz}{\operatorname{ch} z} = 0$.

Within the contour, the poles are where

$$e^{z} + e^{-z} = 0 \quad \text{or} \quad e^{2z} = -1$$

or

$$2z = (2n+1)\pi i$$

$$z = \left(\frac{2n+1}{2}\right)\pi i$$

Only one of these points lies within the contour.

$$z = \frac{\pi}{2} i .$$

So we expand

$$e^{z} + e^{-z} = e^{\frac{\pi}{2}i} e^{z - \frac{\pi}{2}i} + e^{-\frac{\pi i}{2}} e^{-(z - \frac{\pi i}{2})}$$

$$= i\left(1 + z - \frac{\pi}{2}i + \cdots\right) - i\left(1 - z + \frac{\pi i}{2} + \cdots\right)$$

$$= 2i\left(z - \frac{\pi i}{2}\right) + \cdots$$

So the whole contour integral is

$$\int_{\square} \frac{z^2\,dz}{e^{z} + e^{-z}} = \oint \frac{z^2\,dz}{e^{z} + e^{-z}} \quad \text{around } z = \frac{\pi i}{2}$$

$$z = \epsilon e^{i\theta} + \frac{i\pi}{2} \qquad dz = i\epsilon e^{i\theta}\, d\theta$$

$$\oint \frac{z^2\, dz}{2i\,(z - \frac{\pi i}{2})} = \frac{2\pi i}{2i} \left(\frac{\pi i}{2}\right)^2 = -\frac{\pi^3}{4}$$

Similarly, the integral

$$\int_{-R}^{R} \frac{dz}{e^z + e^{-z}} + \int_{R+i\pi}^{-R+i\pi} \frac{dz}{e^z + e^{-z}} = \int \frac{dz}{e^z + e^{-z}}$$

$$= \oint_{z = \frac{\pi i}{2}} \frac{dz}{2i\,(z - i\frac{\pi}{2})} = \frac{2\pi i}{2i} = \pi$$

$$= \int_{-R}^{R} \frac{dz}{e^z + e^{-z}} - \int_{-R}^{R} \frac{dz}{-e^z - e^{-z}} = 2\int_{-R}^{R} \frac{dz}{e^z + e^{-z}}.$$

So the fourth term in Eq. ($*$) is

$$-\pi^2 \int_{-R}^{R} \frac{dx}{e^x + e^{-x}} = -\pi^2 \frac{\pi}{2} = -\frac{\pi^3}{2}.$$

So Eq. ($*$) says

$$-\frac{\pi^3}{4} = 2I - \frac{\pi^3}{2} \quad \text{or} \quad I = \frac{1}{2}\pi^3\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \pi^3/8$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^2 \, dx}{e^x + e^{-x}} = \int_0^{\infty} \frac{(\ln x)^2 \, dx}{1+x^2} = \frac{\pi^3}{8}.$$

Dispersion Relations

The real part of the complex index n of refraction is related to the speed of light in the medium, while the imaginary part is related to the absorption or extinction of the light. Kronig & Kramers in 1926-27 expressed the real part of n^2-1 as an integral of its imaginary part.

Let us consider a function $f(z)$ that is analytic in the UHP and on the real axis, and that vanishes as

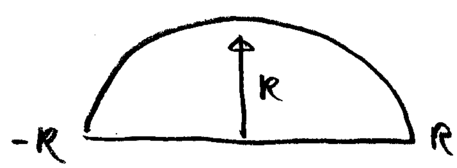
$$\lim_{R\to\infty} |f(Re^{i\theta})| = 0 \quad \text{for} \quad 0 \le \theta \le \pi.$$

So for $z_0 = x_0 + i y_0$ with $y_0 > 0$

we have

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z - z_0}.$$

If the contour C runs along the real axis and then over a great semi-circle of radius R,



, $z = Re^{i\theta}$,

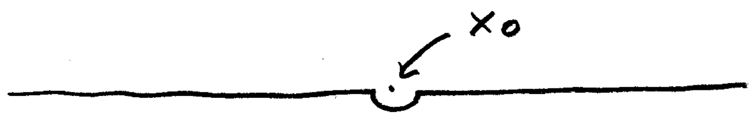
then
$$\left| \int_\Gamma \frac{f(z)\,dz}{z - z_0} \right| \lesssim |f(Re^{i\theta})| \frac{2\pi R}{R} \to 0$$
as $R \to \infty$.

So

$$f(z_0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x - z_0}.$$ We let $z_0 \to x_0$, i.e., $y_0 \to 0$

$$f(x_0) = \frac{1}{2\pi i} \int_{-\infty}^{-\epsilon} \frac{f(x)\,dx}{x - x_0} + \frac{1}{2\pi i} \int_{\epsilon}^{\infty} \frac{f(x)\,dx}{x - x_0}$$
$$+ \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,dz}{z - x_0}$$

The contour now is



Recall that $f(z)$ is analytic in a neighborhood of x_0, and of every real point.

The first two terms are the Cauchy principal value. The third term is

$$\frac{1}{2\pi i}\int_{\smile} \frac{f(z)\,dz}{z - x_0} = \frac{\pi i}{2\pi i} f(x_0) = \underline{\underline{\frac{f(x_0)}{2}}}$$

So

$$f(x_0) = \frac{1}{2\pi i}\, P\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,dx + \frac{f(x_0)}{2} \qquad \text{or}$$

$$f(x_0) = \frac{1}{\pi i}\, P\int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\,dx\,.$$

We let

$$f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$$

$$= u(x) + i\, v(x).$$

Then

$$u(x_0) + i\,v(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \left(\frac{u(x)+i\,v(x)}{i}\right)\frac{dx}{x-x_0}$$

$$= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)\,dx}{x-x_0} - \frac{i}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)\,dx}{x-x_0}$$

or

$$u(x_0) = \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{v(x)}{x-x_0}\,dx$$

and

$$v(x_0) = -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{u(x)\,dx}{x-x_0} .$$

These equations are called <u>dispersion relations</u>. The parts $u(x)$ and $v(x)$ are also said to be <u>Hilbert transforms</u> of each other.

We often need a symmetry relation like

$$f(-x) = f^*(x)$$

to make sense of u and v for $x<0$.

If

$$f(-x) = u(-x) + i\,v(-x) = f^*(x) = u(x) - i\,v(x),$$

then

$$u(-x) = u(x) \qquad \text{even}$$

and

$$v(-x) = -v(x). \qquad \text{odd}$$

Now

$$u(x_0) = \frac{1}{\pi} P\int_{-\infty}^{0} \frac{v(x)}{x-x_0}\,dx + \frac{1}{\pi} P\int_{0}^{\infty} \frac{v(x)\,dx}{x-x_0}$$

$$= \frac{1}{\pi} P\int_{-\infty}^{0} \frac{-v(-x)\,dx}{x-x_0} + \frac{P}{\pi}\int_{0}^{\infty} \frac{v(x)\,dx}{x-x_0}$$

$$y = -x \qquad dx = -dy$$

$$u(x_0) = \frac{1}{\pi} P\int_{\infty}^{0} \frac{v(y)\,dy}{-y-x_0} + \frac{P}{\pi}\int_{0}^{\infty} \frac{v(x)\,dx}{x-x_0}$$

$$= \frac{P}{\pi}\int_{0}^{\infty} \frac{v(y)\,dy}{y+x_0} + \frac{P}{\pi}\int_{0}^{\infty} \frac{v(x)\,dx}{x-x_0}$$

So

$$u(x_0) = \frac{P}{\pi} \int_0^\infty \left(\frac{1}{x+x_0} + \frac{1}{x-x_0} \right) v(x)\,dx$$

$$= \frac{2}{\pi} P \int_0^\infty \frac{x\, v(x)}{x^2 - x_0^2}\,dx .$$ Similarly,

$$v(x_0) = -\frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x)\,dx}{x - x_0}$$ becomes

$$v(x_0) = -\frac{P}{\pi} \int_{-\infty}^{0} \frac{u(-x)}{x-x_0}\,dx \quad - \frac{P}{\pi} \int_0^\infty \frac{u(x)}{x-x_0}\,dx$$

$$= \frac{P}{\pi} \int_\infty^0 \frac{u(y)\,dy}{-y-x_0} \quad - \frac{P}{\pi} \int_0^\infty \frac{u(x)}{x-x_0}\,dx$$

$$= \frac{P}{\pi} \int_0^\infty \left(\frac{1}{x+x_0} - \frac{1}{x-x_0} \right) u(x)\,dx$$

$$= -\frac{2}{\pi} x_0 P \int_0^\infty \frac{u(x)}{x^2 - x_0^2}\,dx .$$

Optical Dispersion

$e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ $v = \omega/k$, $n = c/v = ck/\omega$

If ϵ is the electric permittivity and σ the conductivity, then

$$k^2 = \epsilon \frac{\omega^2}{c^2}\left(1 + i\,\frac{4\pi\sigma}{\omega\epsilon}\right)$$

cgs units with $\mu = 1$. For $4\pi\sigma/\omega\epsilon << 1$

$$k = \sqrt{\epsilon}\,\frac{\omega}{c} + i\,\frac{2\pi\sigma}{c\sqrt{\epsilon}}$$

$$e^{i(kx-\omega t)} = e^{i\omega(x\sqrt{\epsilon}/c - t)}\, e^{-2\pi\sigma x/c\sqrt{\epsilon}}$$

$$n^2 = \frac{c^2 k^2}{\omega^2} = \epsilon + i\,\frac{4\pi\sigma}{\omega}$$

Use $f(\omega) = n^2(\omega) - 1 \rightarrow 0$ as $|\omega| \rightarrow \infty$.

Then K-K had.

$$\mathrm{Re}\left(n^2(\omega_0) - 1\right) = \frac{2}{\pi} P\int_0^\infty \frac{\omega\, \mathrm{Im}(n^2(\omega))}{\omega^2 - \omega_0^2}\, d\omega$$

$$\mathrm{Im}\left(n^2(\omega_0)\right) = -\frac{2}{\pi}\,\omega_0\, P\int_0^\infty \frac{\mathrm{Re}(n^2(\omega) - 1)}{\omega^2 - \omega_0^2}\, d\omega.$$

Parseval's Relation

We saw that for functions $f(z)$ analytic in the UMP and on the real axis, if

$$\lim_{r\to\infty} |f(re^{i\theta})| \to 0 \quad \text{for} \quad 0 \le \theta \le \pi,$$

then

$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0}\, dx . \qquad \text{A.W. } (7.79)$$

So if $f(z)$ is such a function, we may write

$$f(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t - x}\, dx \qquad \text{(with P understood)}$$

and using (7.79) again

$$f(t) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{dx}{t - x} \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(s)\, ds}{s - x}$$

$$= \int_{-\infty}^{\infty} ds\, f(s) \int_{-\infty}^{\infty} \frac{dx}{\pi^2 (t - x)(s - x)} \qquad \text{whence}$$

$$\delta(t - s) = \frac{1}{\pi^2} P \int_{-\infty}^{\infty} \frac{dx}{(t - x)(s - x)}$$

for such analytic functions $f(z)$.

Delta functions are defined for specific classes of "test" functions — the functions $f(x)$ for which they work.

Now suppose that $u(z)$ and $v(z)$ are the real and imaginary parts of such analytic functions, related by the Hilbert relations

$$u(x_0) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0} \, dx \tag{7.81 a}$$

and

$$v(x_0) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0} \, dx . \tag{7.81 b}$$

Suppose also that the integrals

$$\int_{-\infty}^{\infty} |u(x)|^2 dx \quad \text{and} \quad \int_{-\infty}^{\infty} |v(x)|^2 dx$$

are well defined. Then, using (7.81a) twice, we get

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} dx \, \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(s)\, ds}{s - x} \; \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{v(t)\, dt}{t - x}$$

$$= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \;\; \delta(t-s) \, v(s) \, v(t)$$

$$= \int_{-\infty}^{\infty} |v(s)|^2 \, ds .$$

Causality requires the effect $H(t)$

$$H(t) = \int_{-\infty}^{\infty} F(t-t')\, G(t')\, dt'$$

to occur later than its cause $G(t')$.
So we set

$$F(t-t') = 0 \quad \text{if} \quad t < t'.$$

If

$$h(\omega) = \int_{-\infty}^{\infty} dt\, H(t)\, \frac{e^{i\omega t}}{\sqrt{2\pi}}$$

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, F(t)\, e^{i\omega t}$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, G(t)\, e^{i\omega t}, \quad \text{then}$$

$$h(\omega) = \int_{-\infty}^{\infty} dt\, H(t)\, \frac{e^{i\omega t}}{\sqrt{2\pi}} = \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}}\, e^{i\omega t} \int_{-\infty}^{\infty} dt'\, F(t-t')\, G(t').$$

Well

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, g(\omega)\, e^{-i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dt'\, G(t')\, e^{i\omega(t'-t)}$$

So if we accept that the representation

$$\delta(t'-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; e^{i\omega(t'-t)}$$

is appropriate for square-integrable functions, then

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \; g(\omega)\, e^{-i\omega t} = \int_{-\infty}^{\infty} dt' \, G(t') \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, e^{i\omega(t'-t)}$$

$$= \int_{-\infty}^{\infty} dt' \, G(t') \, \delta(t'-t) = G(t).$$

So

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \; g(\omega)\, e^{-i\omega t}.$$

Similarly,

$$F(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \; f(\omega)\, e^{-i\omega t}$$

and

$$H(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \; h(\omega)\, e^{-i\omega t}.$$

So

$$h(\omega) = \int_{-\infty}^{\infty} dt \, \frac{e^{i\omega t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\omega' \, f(\omega') \frac{e^{-i\omega'(t-t')}}{\sqrt{2\pi}}$$

$$\times \int_{-\infty}^{\infty} d\omega'' \, \frac{e^{-i\omega'' t'}}{\sqrt{2\pi}} \, g(\omega'')$$

Now using twice

$$\delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{-it(\omega-\omega')} ,$$ we get

$$h(\omega) = \sqrt{2\pi} \int_{-\infty}^{\infty} d\omega' \, d\omega'' \, \delta(\omega-\omega') \, \delta(\omega'-\omega'') f(\omega') g(\omega'')$$

$$= \sqrt{2\pi} \; f(\omega) \, g(\omega) .$$

So the Fourier transform of the convolution

$$H(t) = \int_{-\infty}^{\infty} F(t-t') G(t') dt'$$ is the product

$$h(\omega) = \sqrt{2\pi} \; f(\omega) \, g(\omega)$$ of the Fourier transforms.

Titchmarsh: If $f(\omega)$ is square integrable on the real ω-axis, then any one of these three statements implies the other two:

(1) The Fourier transform $F(t)$ of $f(\omega)$ is zero for $t<0$. (This is causality.)

(2) $f(z)$ is analytic in the UHzP for $y>0$ $(\omega = z = x+iy)$ and approaches $f(x)$ almost everywhere as $y \to 0$. Also its norm is bounded

$$\int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx < K \qquad \text{for } y>0.$$

(3) If $f = u + iv$, then u and v are Hilbert transforms of each other, Eqs. (7.81a-b).

Well, (1) implies that

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \; F(t) \, e^{izt}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dt \; F(t) \, e^{izt}$$

is surely analytic in the UHzP. And $f(z)$ becomes smaller as y increases. And we have seen that under such conditions u & v are related by Hilbert transforms.

Method of Steepest Descent

Consider the integral

$$I(s) = \int_C g(z)\, e^{s f(z)}\, dz,$$

where $f(z)$ and $g(z)$ are analytic functions, as the real variable s increases.

We may vary the contour C as long as we collect the residues of any poles that we cross.

This integral will be dominated by the real part of $f(z) = u(z) + i v(z)$. By the C-R conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}.$$

So the real part u of an analytic function is <u>harmonic</u>:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

Also

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Since u is harmonic, it can not have any true minima. Instead if

$$\frac{\partial u(z_0)}{\partial x} = 0 = \frac{\partial u(z_0)}{\partial y}, \quad \text{then}$$

$u(z_0) = u(x_0, y_0)$ has a saddle point at z_0. That is, u has a maximum along one curve and a minimum along another curve, both at $z_0 = x_0 + i y_0$.

If $f(z)$ has several saddle points, then we must cope with each of them.

Let's look at one at z_0. Then

$$f'(z_0) = 0 \quad \text{and near} \quad z = z_0$$

$$f(z) = f(z_0) + \frac{1}{2}(z - z_0)^2 f''(z_0).$$

Say $\quad f''(z_0) = \rho\, e^{i\phi}$. Then we choose our contour thru z_0 to satisfy $z - z_0 = x e^{i\theta}$

$$(z - z_0)^2 = x^2 e^{2i\theta} \qquad \text{with}$$

$$e^{i\phi} e^{2i\theta} = -1$$

So $\quad \phi + 2\theta = \pi \quad$ and $\quad \phi + 2(\theta - \pi) = \pi$

will describe the straight line thru z_0.
So near z_0

$$f(z) = f(z_0) + \tfrac{1}{2} x^2 e^{2i\theta} \rho e^{i\phi}$$

$$= f(z_0) - \tfrac{1}{2} x^2 \rho \,. \qquad \text{So} \quad z = x e^{i\theta}$$

and

$$dz = e^{-i\theta} dx$$

$$I(s) \cong e^{i\theta} \int_{-\infty}^{\infty} dx \; g(z_0)\, e^{s f(z_0) - \frac{1}{2} s \rho x^2}$$

$$= e^{i\theta} g(z_0)\, e^{s f(z_0)} \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2} s \rho x^2}$$

$$= e^{i\theta} g(z_0)\, e^{s f(z_0)} \sqrt{\frac{2}{s\rho}} \int_{-\infty}^{\infty} dx \, e^{-x^2}$$

$$= \sqrt{\frac{2\pi}{s\rho}}\; g(z_0)\, e^{s f(z_0)}\, e^{i\theta}$$

$$= \sqrt{\frac{2\pi}{s \rho e^{-2i\theta}}}\; g(z_0)\, e^{s f(z_0)}$$

Now

$$\rho e^{-2i\theta} = \rho e^{i\phi - i\pi} = -\rho e^{i\phi} = -f''(z_0),$$

So

$$I(s) = \sqrt{\frac{2\pi}{-s\, f''(z_0)}}\; g(z_0)\, e^{s f(z_0)} .$$