The inverse function is

$$w = \ln z = u + iv = \ln re^{i\theta}$$

$$re^{i\theta} = z = e^{w} = e^{u+iv}$$

So $\quad r = e^{u}$ and $\theta = v$

or

$$u = \ln r = \ln |z| \qquad \text{and}$$

$$v = \theta = \arg z$$

So

$$\ln z = \ln |z| + i \arg z .$$

But $\arg z$ is multi-valued.

$$z = e^{u+iv+2\pi i n} \qquad n \text{ an integer.}$$

To have a one-to-one $\ln z$, we must define & restrict $\arg z$. Two popular choices are:

$$0 \le \arg z < 2\pi \qquad \text{cut on } (0, \infty)$$

and

$$-\pi < \arg z \le \pi . \qquad \text{cut on } (-\infty, 0).$$

Consider how $dz = \epsilon e^{i\theta}$ and $dz' = \epsilon e^{i\theta'}$ are mapped by the analytic function

$$w = f(z).$$

Clearly

$$dw = f'(z)\,dz = f'(z)\,\epsilon e^{i\theta}$$

and

$$dw' = f'(z)\,dz' = f'(z)\,\epsilon e^{i\theta'}$$

So the angle between $dw = \rho e^{i\phi}$ and

$$dw' = \delta e^{i\phi'}$$

is the argument of their ratio

$$\frac{dw'}{dw} = \frac{f'(z)\,\epsilon e^{i\theta'}}{f'(z)\,\epsilon e^{i\theta}} = e^{i(\theta'-\theta)} = \frac{\delta e^{i\phi'}}{\delta e^{i\phi}} = e^{i(\phi'-\phi)}$$

So

$$e^{i(\phi'-\phi)} = e^{i(\theta'-\theta)}.$$

So

$$\phi' - \phi = \theta' - \theta \qquad \text{mod } 2\pi,$$

Angles are preserved under analytic or <u>conformal</u> maps.

A function that is analytic in the (finite) complex plane apart from isolated poles is <u>meromorphic</u> (part form).

Entire functions are meromorphic functions with no poles at all.

e^z is entire

$$\frac{2}{e^z + e^{-z}} = \cosh z \quad \text{is meromorphic}$$

with poles at $z = (2n+1)\frac{\pi}{2} i$.

If the Laurent expansion of $f(z)$ about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

runs from $n = -m$ to $n = \infty$, then $f(z)$ has a pole of order m at z_0.

But if the sum really runs from $n = -\infty$ to $n = \infty$, then $f(z)$ has an essential singularity at $z = z_0$.

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{z}\right)^n$$

$$= \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

has an essential singularity at $z=0$.

Problem 7.1.5 For what z's is

$$e^{\frac{1}{z}} = z_0 \neq 0 ?$$

$$\frac{1}{z} = \log z_0 + 2\pi n i$$

$$z = \frac{1}{\log z_0 + 2\pi n i} .$$

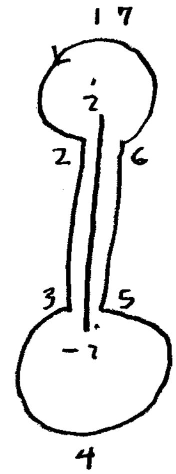
There are infinitely many solutions.

Problem 7.1.3 $f(z) = (z^2+1)^{\frac{1}{2}}$

$$f(z) = (z-i)^{1/2} (z+i)^{1/2}$$

Let $z-i = re^{i\theta}$ and $z+i = \rho e^{i\phi}$.

$$f(z) = \sqrt{r\rho}\, e^{i(\theta+\phi)/2} \qquad \frac{\pi}{2} \leq \theta,\phi \leq 2\pi + \frac{\pi}{2}$$



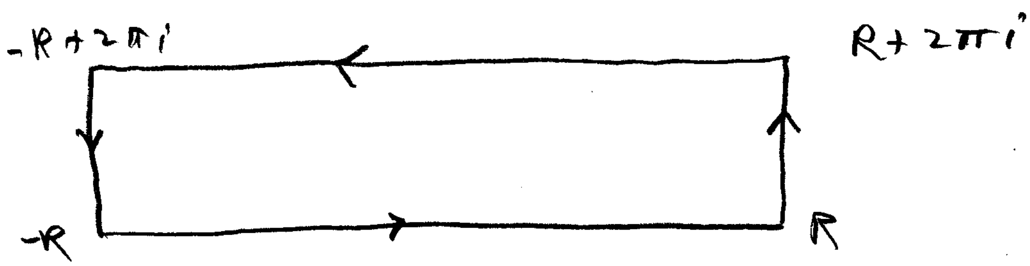
The cut line runs from $-i$ to i. So defined, $f(z)$ is single valued apart from the cut, across which the discontinuity in phase is π.

Point	θ	ϕ	$(\theta+\phi)/2$
1	$\pi/2$	$\pi/2$	$\pi/2$
2	$3\pi/2$	$\pi/2$	π
3	$3\pi/2$	$\pi/2$	π
4	$3\pi/2$	$3\pi/2$	$3\pi/2$
5	$3\pi/2$	$2\pi+\pi/2$	2π
6	$3\pi/2$	$2\pi+\pi/2$	2π
7	$2\pi+\pi/2$	$2\pi+\pi/2$	$2\pi+\pi/2$

Let's evaluate

$$I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx \quad \text{for} \quad 0 < a < 1.$$

We choose the contour



the short segments vanish because as $R \to \infty$

$$\left| \frac{e^{az}}{1+e^{z}} \right| \approx e^{-(a+1)R} \to 0$$

and as $R \to -\infty$

$$\left| \frac{e^{az}}{1+e^{z}} \right| \approx e^{(-1+a)R} \to 0, \text{ since } a<1$$

Now on both long segments e^{z} is the same, since

$$e^{x} = e^{x+2\pi i}.$$

So

$$J = \int_{R+2\pi i}^{-R+2\pi i} \frac{e^{az}}{1+e^{z}}\,dz = e^{2\pi i a} \int_{R}^{-R} \frac{e^{ax}\,dx}{1+e^{x}}$$

$$= -e^{2\pi i a} I.$$ So the contour integral is

$$I+J = (1-e^{2\pi i a})I = \oint \frac{e^{az}}{1+e^{z}}\,dz.$$

The function $e^{az}/1+e^{z}$ has a pole at

$$e^{z} = -1 \qquad \text{or} \qquad z = i\pi \,.$$

Near that pole

$$e^{z} = e^{i\pi} e^{z-i\pi}$$

$$= -e^{z-i\pi}$$

$$= -(1 + z - i\pi)$$

So integrating around $z = i\pi$ in a tiny circle, we get

$$\oint \frac{e^{az}}{1+e^{z}} dz = e^{i\pi a} \oint \frac{dz}{1-1-z+i\pi}$$

$$= e^{i\pi a} \oint \frac{dz}{-z+i\pi} = -e^{i\pi a} \oint \frac{dz}{z-i\pi}$$

$$= -2\pi i \, e^{i\pi a} = (1 - e^{2\pi i a}) I$$

whence

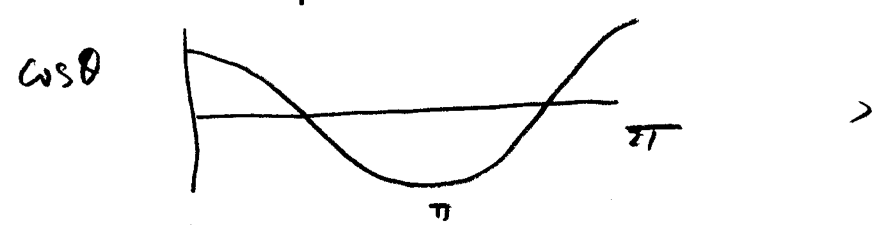
$$I = -\frac{2\pi i \, e^{i\pi a}}{(1-e^{2\pi i a})} = \pi \frac{2i}{e^{i\pi a} - e^{-i\pi a}} = \frac{\pi}{\sin \pi a} \,.$$

Problem 7.2.8

$$I = \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} \qquad \text{for} \qquad a>1.$$

Since $a>1$, there is no singularity on the original contour, $\theta: 0 \to \pi$.

Since $\cos\theta$ repeats on $\theta: \pi \to 2\pi$



we have

$$I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2}.$$

Let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta = iz\, d\theta$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Then

$$I = \frac{1}{2i} \oint \frac{dz}{z\left[a + \frac{1}{2}\left(z+\frac{1}{z}\right)\right]^2}$$

$$= \frac{2}{i} \oint \frac{z\, dz}{\left[2az + z^2 + 1\right]^2}$$

Where are z poles?

$$0 = z^2 + 2az + 1$$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2}$$

$$= -a \pm \sqrt{a^2-1}$$

$$z_+ = \sqrt{a^2-1} - a = a\left(\sqrt{1-\frac{1}{a^2}} - 1\right)$$

$$0 > z_+ \stackrel{\sim}{=} a\left(1 - \frac{1}{2a^2} - 1\right) \approx -\frac{1}{2a} > -\frac{1}{2}$$

So this pole lies within the contour $|z|=1$.

$$z_- = -a - \sqrt{a^2-1} = -a\left(1 + \sqrt{1-\frac{1}{a^2}}\right)$$

$$\approx -a\left(1 + 1 - \frac{1}{2a^2}\right)$$

$$z_- \approx -2a + \frac{1}{2a} < -2 + \frac{1}{2} = -\frac{3}{2}$$

So z_- lies outside the circle $|z|=1$.

So

$$I = \frac{2}{i} \oint \frac{z\,dz}{(z-z_-)^2(z-z_+)^2}$$

Recall (6.47):

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^{n+1}} .$$

So

$$I = \frac{2}{i}\, 2\pi i \, \frac{d}{dz} \frac{z}{(z-z_-)^2} \Bigg|_{z=z_+}$$

$$= 4\pi \left[\frac{1}{(z_+ - z_-)^2} - \frac{2z_+}{(z_+ - z_-)^3} \right]$$

$$z_+ - z_- = 2\sqrt{a^2-1} \qquad \text{So}$$

$$I = 4\pi \left[\left(\frac{1}{2\sqrt{a^2-1}} \right)^2 - \frac{2(-a+\sqrt{a^2-1})}{(2\sqrt{a^2-1})^3} \right]$$

$$= 4\pi \left[\frac{2\sqrt{a^2-1} - 2(-a+\sqrt{a^2-1})}{8\,(a^2-1)^{3/2}} \right]$$

$$= \frac{\pi a}{(a^2-1)^{3/2}}, \qquad \text{for } a > 1.$$

$$I = \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$$

Assume $f(z)$ analytic in UHzP except for a finite number of poles and

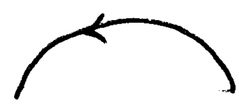
$$\lim_{|z|\to\infty} f(z) = 0 \quad \text{for } 0 \le \arg z \le \pi .$$

then

$$\int_{-\infty}^{\infty} f(x) e^{iax}\, dx = \oint f(z)\, e^{iaz}\, dz$$

$$= 2\pi i \sum \text{residues in UHzP for } a > 0;$$

$$= -2\pi i \sum \text{residues in LHzP for } a < 0.$$

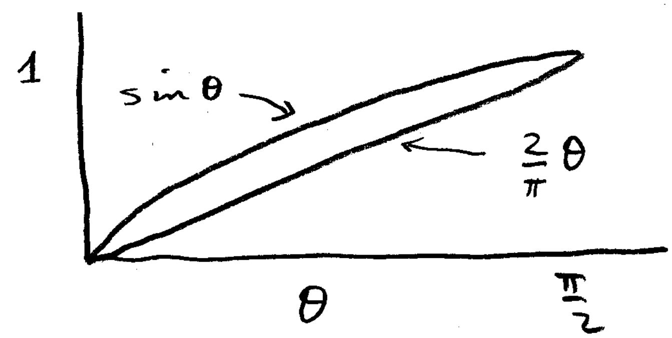
On the added contour  for $a > 0$

$$I_R = \int_{\curvearrowleft} f(z)\, e^{iaz}\, dz$$

$$= \int_0^{\pi} F(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta$$

$$|I_R| \le \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta = 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta .$$

Now from the figure



,

we see that

$$\frac{2}{\pi}\theta \leqslant \sin\theta \quad \text{for} \quad 0 \le \theta \le \frac{\pi}{2}.$$

So

$$|I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-aR\frac{2\theta}{\pi}}\, d\theta$$

$$= 2\epsilon R \left[\frac{-\pi e^{-aR\frac{2\theta}{\pi}}}{aR2} \right]_0^{\pi/2}$$

$$= 2\epsilon R \left[-\frac{\pi}{2}\frac{e^{-aR}}{aR} + \frac{\pi}{2aR} \right]$$

$$= \frac{\pi}{a}\epsilon\left(1 - e^{-aR}\right) \to \frac{\pi\epsilon}{a} \quad \text{as } R \to \infty.$$

$$\to 0 \quad \text{as } \epsilon \to 0.$$

So

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx + \lim_{R\to\infty} I_R = 2\pi i \sum_{UHP} \text{residues}$$

or since $I_R \to 0$ as $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum_{UHP} \text{residues of } f(z).$$

Problem 7.2.14(a) With $k > 0$

$$I = \int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx$$

$= I_+ + I_-$. But $I_+ = I_-$, so

$$I = \int_{-\infty}^{\infty} \frac{e^{ikx}}{x^2+a^2}\, dx = \oint \frac{e^{ikz}}{(z+ia)(z-ia)}\, dz$$

$$= 2\pi i \, \frac{e^{ikia}}{2ia} = \frac{\pi}{a} e^{-ka}.$$

Problem 7.2.15

$$I = \int_{-\infty}^{\infty} \frac{\sin kx}{x}\, dx \qquad \text{for } k>0$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ikx} - e^{-ikx}}{x}\, dx = I_+ + I_-$$

$$I_+ = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{ikx}\, dx}{x}$$

$$I_- = -\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-ikx}}{x}\, dx \qquad \begin{array}{l} -x = y \\ dx = -dy \end{array}$$

$$= -\frac{1}{2i}\int_{\infty}^{-\infty} \frac{e^{iky}}{-y}(-dy) = -\frac{1}{2i}\int_{\infty}^{-\infty} e^{iky}\frac{dy}{y}$$

$$= \frac{1}{2i}\int_{-\infty}^{\infty} e^{iky}\frac{dy}{y} = I_+$$

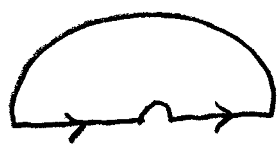
So

$$I = 2I_+ = \frac{1}{i}\int_{-\infty}^{\infty} \frac{e^{ikx}}{x}\, dx$$

We interpret this as a principal-value integral.

$$I = \frac{1}{i}\int_{-\infty}^{-\epsilon} \frac{e^{ikx}}{x}\,dx + \frac{1}{i}\int_{\epsilon}^{\infty} \frac{e^{ikx}}{x}\,dx$$

$$= \frac{1}{i}\int \frac{e^{ikz}}{z}\,dz - \frac{1}{i}\int_{\frown} \frac{e^{ikz}}{z}\,dz$$



$$= -\frac{1}{i}\int_{\frown} e^{ikz}\,\frac{dz}{z} \qquad z = \epsilon e^{i\theta}$$

$$dz = i\epsilon e^{i\theta}\,d\theta$$

$$= -\frac{1}{i}\int_{\pi}^{0} e^{ikz}\,\frac{i\epsilon e^{i\theta}\,d\theta}{\epsilon e^{i\theta}}$$

$$= \int_{0}^{\pi} d\theta = \pi \qquad \text{for} \quad k>0$$

Do case $k<0$ for homework.

If $k=0$, the principal-value integral

$$I = \frac{1}{i}\int_{-\infty}^{-\epsilon} \frac{dx}{x} + \frac{1}{i}\int_{\epsilon}^{\infty} \frac{dx}{x} = 0.$$