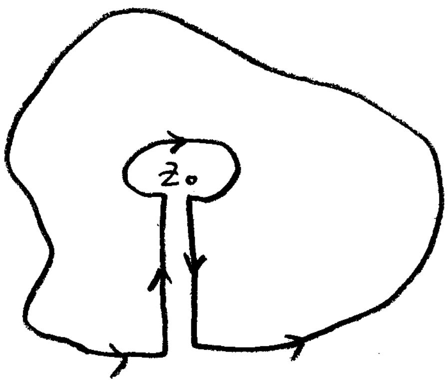
Now the integral along C_1 is



which is a closed curve in a region with no holes in which $f(z)/(z-z_0)$ analytic.

So
$$\oint_{C_1} \frac{f(z)}{z-z_0} = 0.$$

The integral along C_2 just cancels the clockwise integral around z_0. So

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)\, dz}{z-z_0}$$

in which C is the more-or-less arbitrary counter-clockwise contour around z_0 and C_2 is a tight c-cw circle around z_0. So

$$\frac{1}{2\pi i} \oint_C \frac{f(z)\, dz}{z-z_0} = \frac{1}{2\pi i} \int_0^{2\pi} d\theta \, \frac{[f(z_0) + f'(z_0)\, re^{i\theta}]\, ire^{i\theta}}{re^{i\theta}}$$

where $z - z_0 = re^{i\theta}$ $\quad z = z_0 + re^{i\theta}$

$dz = ire^{i\theta}\, d\theta.$

So

$$\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{z-z_0} = \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[f(z_0) + f'(z_0) r e^{i\theta} \right]$$

$$= f(z_0).$$

This is Cauchy's integral formula:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

in which C is a c-c-w contour around z_0 that lies within a hole-free region of analyticity for $f(z)$.

$$\frac{f(z_0+dz_0) - f(z_0)}{dz_0} = \frac{1}{2\pi i\, dz_0}\oint f(z)\,dz \left(\frac{1}{z-z_0-dz_0} - \frac{1}{z-z_0} \right)$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0-dz_0)(z-z_0)}$$

So taking the limit $dz_0 \to 0$, we get

$$f'(z_0) = \frac{1}{(2\pi i)}\oint \frac{f(z)\,dz}{(z-z_0)^2}.$$ Next

$$f^{(2)}(z_0) = \frac{2}{(2\pi i)}\oint \frac{f(z)\,dz}{(z-z_0)^3}$$

In general

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

So if $f(z)$ is analytic in a region, it is also infinitely differentiable there.

If $f(x,y)$ is continuous in a region and every closed integral there vanishes,

$$0 = \oint f(x,y)\,dz,$$

then $f(z) = f(x,y)$ is analytic there.

$$F(z) = \int_{z_0}^{z} f(z)\,dz$$

is independent of the contour and also

$$\frac{dF(z)}{dz} = f(z).$$

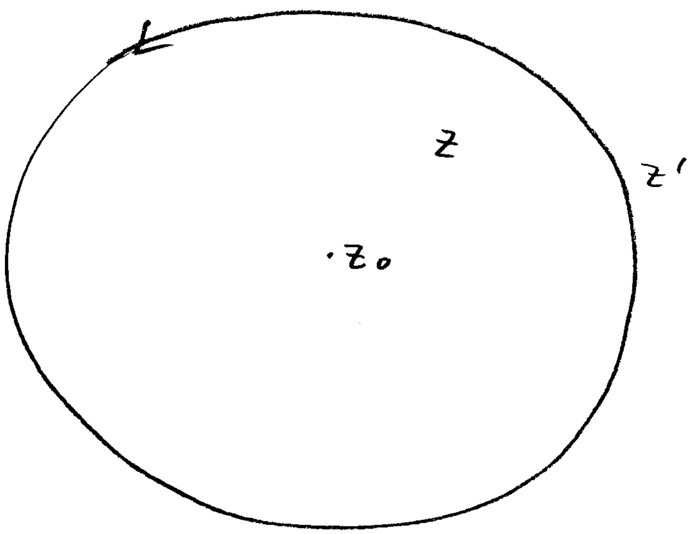
So $F(z)$ is analytic there, Thus $f(z)$ is too.

Taylor expansion

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z_0 - (z-z_0)}$$

$$= \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z_0 \left[1 - \frac{z-z_0}{z'-z_0}\right]}$$



Now $\left|\frac{z-z_0}{z'-z_0}\right| < 1$ so

$$\frac{1}{1 - \frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n$$

converges absolutely and uniformly.

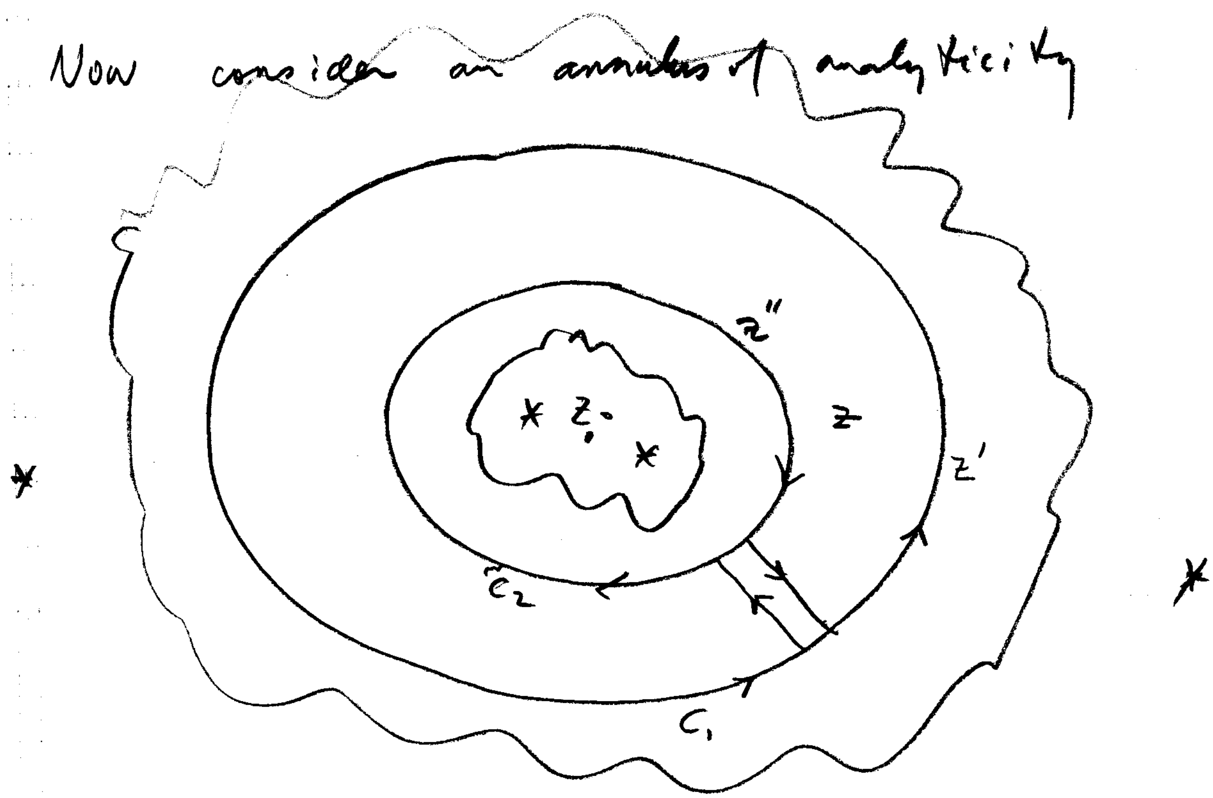
So we may integrate the series

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(z')\,dz'}{z'-z_0} \sum \left(\frac{z-z_0}{z'-z_0}\right)^n$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0)$$

Now consider an annulus of analyticity



and a closed contour C which is equivalent to a c-c-w contour C_1 and a c-w contour $\tilde{C}_2$. Let $C_2 = -\tilde{C}_2$ be the c-c-w version of $\tilde{C}_2$.

Then

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z'')\,dz''}{z''-z}$$

Now if z_0 is the center of both circular contours C_1 and C_2, then

$$\left|\frac{z''-z_0}{z-z_0}\right| < 1 \text{ on } C_2 \;\&\; \left|\frac{z-z_0}{z'-z_0}\right| < 1 \quad \text{on } C_1$$

So

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z'-z_0-(z-z_0)} + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z'')\,dz''}{z-z_0-(z''-z_0)}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]} + \frac{1}{2\pi i}\oint_{C_2} \frac{f(z'')\,dz''}{(z-z_0)\left[1-\frac{z''-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(z'-z_0)^{n+1}} f(z')\,dz' + \frac{1}{2\pi i}\oint_{C_2} \sum_{n=0}^{\infty} \frac{(z''-z_0)^n}{(z-z_0)^{n+1}} f(z'')\,dz''$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$+ \sum_{n=1}^{\infty} \frac{1}{(z-z_0)^n} \frac{1}{2\pi i}\oint_{C_2} f(z'')\,dz''\,(z''-z_0)^{n-1}$$

Now the functions

$$\frac{f(z')}{(z'-z_0)^{n+1}} \quad \text{and} \quad f(z'')(z''-z_0)^{m-1}$$

are analytic in the annulus, so we may displace the contours to a common contour C:

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$

$$+ \sum_{m=-\infty}^{-1} (z-z_0)^m \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)^{m+1}}$$

Let

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}},$$

then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \text{ is}$$

the Laurent series for $f(z)$ in the annulus.

The $*$'s represent possible singularities. If they are merely poles, then $f(z)$ is meromorphic. A pole of order n is

$$\frac{1}{(z-z_*)^n}.$$

Note that much of C.V. theory follows from the vanishing of the integral of an analytic function around a tiny square.

$$0 = \oint_{\square} f(z)\,dz$$

Thus we may shift a contour

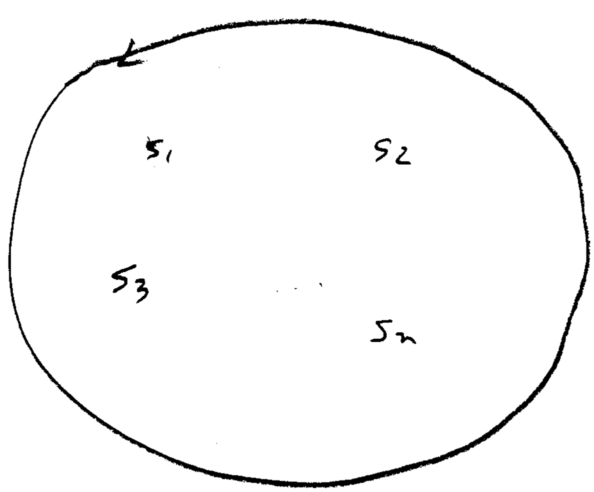
from

to

because $C_2 - C_1 = \square$ and $\oint_{\square} f(z)\,dz = 0$.

By iterating such shifts by one pixel at a time, one may move a contour quite generally throughout a region of analyticity without changing the value of the integral.

Now suppose we have a contour that encloses n isolated singularities in a region in which $f(z)$ is analytic.



Then by shrinking the contour, we get

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz$$

$$= \sum_{k=1}^{m} 2\pi i \, a_{-1}(z_k)$$

in which we've used the moronic notation

$$a_{-1}(z_k) = \frac{1}{2\pi i} \oint_{C_k} f(z)\,dz \, .$$

These integrals $a_{-1}(z_k)$ are called residues.

So

$$\oint_C f(z)\,dz = 2\pi i \sum a_{-1}(z_k)$$

$$= 2\pi i \text{ (sum of enclosed residues)}.$$

This is the residue theorem.