Finite Groups have a finite number $n$ of elements.
$n$ is the order of the group.
A group $G$:

(1) $f, g \in G \Rightarrow fg \in G$

(2) $f(gh) = (fg)h$

(3) $\exists\, e \in G$ ∋ $ef = fe = f \quad \forall f \in G$

(4) if $f \in G$, then $\exists\, f^{-1}$ ∋ $ff^{-1} = f^{-1}f = e$.

A representation of $G$ is a map $D$ onto linear operators ∋

(1) $D(e) = 1$

2 $D(g_1)\, D(g_2) = D(g_1 g_2)$.

E.g.

$Z_2$ $\quad e, a \qquad ea = a \qquad ee = e$

$\qquad ae = a \qquad aa = e$

| | e | a |
|---|---|---|
| e | e | a |
| a | a | e |

If $g_1 g_2 = g_2 g_1 \quad \forall g_1, g_2 \in G$, then

$G$ is abelian.

Rep of $Z_2$: $D(e) = 1 \qquad D(a) = -1$.

This is a 1-D rep of $Z_2$ because the D's act on a 1-D space.

One may associate with each $g \in G$ a vector $|g\rangle$ and let the $\{|g_i\rangle\}$ be an ON basis. Then

$$D(g_1)\,|g_2\rangle = |g_1 g_2\rangle$$

is the regular rep.

Consider, e.g., $Z_3$

| | e | a | b |
|---|---|---|---|
| e | e | a | b |
| a | a | b | e |
| b | b | e | a |

$|e_1\rangle = |e\rangle \qquad |e_2\rangle = |a\rangle \qquad |e_3\rangle = |b\rangle$

$D(g)_{ij} = \langle e_i | D(g) | e_j \rangle$. Then

$D(e)_{ij} = \langle e_i | D(e) | e_j \rangle = \langle e_i | e e_j \rangle = \langle e_i | e_j \rangle$

$= \delta_{ij}$ so

$$D(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

$D(a)_{ij} = \langle e_i | D(a) | e_j \rangle$ so

$D(a)_{11} = \langle e | D(a) | e \rangle = \langle e | a e \rangle = \langle e | a \rangle = \langle e_1 | e_2 \rangle = 0$

$D(a)_{12} = \langle e | D(a) | a \rangle = \langle e | a a \rangle = \langle e | b \rangle = 0$

$D(a)_{13} = \langle e | D(a) | b \rangle = \langle e | a b \rangle = \langle e | e \rangle = 1$

one gets $D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

This is the regular rep of $Z_3$. Another is

$D(e) = 1 \qquad D(a) = e^{2\pi i/3} \qquad D(b) = e^{4\pi i/3}$.

The matrices are matrix elements of the linear operators.

$$[D(g)]_{ij} = \langle e_i | D(g) | e_j \rangle.$$

So

$$[D(g_1 g_2)]_{ij} = [D(g_1) D(g_2)]_{ij}$$

$$= \langle e_i | D(g_1) | e_k \rangle \langle e_k | D(g_2) | e_j \rangle$$

$$= \sum_k D(g_1)_{ik} \, D(g_2)_{kj} \quad .$$

Irreducible Rep..

Similarity transformation

$$D(g) \to D'(g) = S^{-1} D(g) S$$

this is another rep, an equivalent rep.

A rep is unitary if all the $D(g)$'s are unitary $D(g)^\dagger = D(g)^{-1}$ which implies $D_{ij}(g)$ is a unitary matrix.

All reps of finite groups are equivalent to unitary reps.

A rep. is reducible if it has an invariant subspace

$$P\, D(g)\, P = D(g) P \qquad \forall g \in G$$

The reg. rep of $Z_3$ has inv. subspace projected by

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

A rep is irreducible if it is not reducible.

A rep is completely reducible if it is equivalent to a rep like

$$\begin{pmatrix} D_1(g) & 0 & 0 \\ 0 & D_2(g) & 0 \\ 0 & 0 & D_3(g) \end{pmatrix}$$

ie can be put in block-diagonal form.

Kevin Cahill Physics 522 November 19, 2003

A convenient choice of Dirac matrices, used by Weinberg, is

$$\gamma = -i \begin{pmatrix} \mathbf{0} & \sigma \\ -\sigma & \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \gamma^0 = -i\beta = -i \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}. \tag{1}$$

They satisfy the anti-commutation relations

$$[\gamma^a, \gamma^b]_+ = 2\eta^{ab}, \tag{2}$$

in which the flat space-time metric is

$$\eta^{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Under hermitian conjugation, they transform as $\gamma^\dagger = \gamma$ and $(\gamma^0)^\dagger = -\gamma^0$. For this choice of Dirac matrices, we may define Majorana and Dirac fields in terms of the scalar-like lawn

$$\phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}} \left[ \begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p}) e^{ipx} + i \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A^*(\mathbf{p}) e^{-ipx} \right] \tag{4}$$

where $I$ and $\sigma_2$ are the $2 \times 2$ matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5}$$

$A(\mathbf{p})$ and $A^*(\mathbf{p})$ are the 2-vectors

$$A(\mathbf{p}) = \begin{pmatrix} a(\mathbf{p},+) \\ a(\mathbf{p},-) \end{pmatrix} \quad \text{and} \quad A^*(\mathbf{p}) = \begin{pmatrix} a^\dagger(\mathbf{p},+) \\ a^\dagger(\mathbf{p},-) \end{pmatrix}, \tag{6}$$

$p^0 = \sqrt{m^2 + \mathbf{p})^2}$, and $\hbar = c = 1$. The lawn $\phi(x)$ describes a single spin-one-half particle that is its own anti-particle.

Since $m^2 + p^2 = m^2 + \mathbf{p}^2 - (p^0)^2 = 0$, the lawn $\phi(x)$ satisfies the Klein-Gordon equation

$$(m^2 + \partial_0^2 - \nabla^2)\phi(x) = (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) = 0. \tag{7}$$

The Majorana field $\chi(x)$ is obtained from derivatives of the lawn $\phi(x)$:

$$\chi(x) = (m - \gamma^a \partial_a)\beta\phi(x). \tag{8}$$

It automatically satisfies the Dirac equation:

$$\begin{aligned} (\gamma^a\partial_a + m)\,\chi(x) &= (\gamma^a\partial_a + m)\,(m - \gamma^a\partial_a)\,\beta\phi(x) \\ &= \left(m^2 - \gamma^a\gamma^b\partial_a\partial_b\right)\beta\phi(x) \\ &= \left(m^2 - \tfrac{1}{2}[\gamma^a, \gamma^b]_+\partial_a\partial_b\right)\beta\phi(x) \\ &= \left(m^2 - \eta^{ab}\partial_a\partial_b\right)\beta\phi(x) \\ &= \beta\left(m^2 - \eta^{ab}\partial_a\partial_b\right)\phi(x) = 0. \end{aligned}$$

Suppose that there are two spin-one-half particles of the same mass $m$ described by the two operators $a_1(\mathbf{p},\sigma)$ and $a_2(\mathbf{p},\sigma)$ which satisfy the anti-commutation relations

$$[a_i(\mathbf{p},\sigma), a_j^\dagger(\mathbf{p}',\sigma')]_+ = \delta_{\sigma\sigma'}\delta^3(\mathbf{p}-\mathbf{p}'). \tag{9}$$

Then by following Eqs.(4–9) and defining two 2-vectors $A_i(\mathbf{p},\sigma)$ as in (6), we may construct the two lawns

$$\phi_i(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}}\left[\begin{pmatrix} I \\ I \end{pmatrix} A_i(\mathbf{p})e^{ipx} + i\begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A_i^*(\mathbf{p})e^{-ipx}\right] \tag{10}$$

and from them the two Majorana fields

$$\chi_i(x) = (m - \gamma^a\partial_a)\beta\phi_i(x) \tag{11}$$

which satisfy the Dirac equation

$$(\gamma^a\partial_a + m)\,\chi_i(x) = 0. \tag{12}$$

But because the two lawns $\phi_i(x)$ are of the same mass, we may combine them into the complex lawn

$$\Phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right]. \tag{13}$$

From the complex operators

$$a(\mathbf{p},\sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p},\sigma) + ia_2(\mathbf{p},\sigma)\right] \tag{14}$$

and

$$a^c(\mathbf{p},\sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p},\sigma) - ia_2(\mathbf{p},\sigma)\right], \tag{15}$$

we may form the complex 2-vectors

$$A(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) + iA_2(\mathbf{p})\right] = \begin{pmatrix} a(\mathbf{p},+) \\ a(\mathbf{p},-) \end{pmatrix} \tag{16}$$

and

$$A^c(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right] = \begin{pmatrix} a^c(\mathbf{p},+) \\ a^c(\mathbf{p},-) \end{pmatrix}. \tag{17}$$

The complex lawn involves $A(\mathbf{p})$ and

$$A^{c*}(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right]^* = \frac{1}{\sqrt{2}}\left[A_1^*(\mathbf{p}) + iA_2^*(\mathbf{p})\right] = \begin{pmatrix} a^{c*}(\mathbf{p},+) \\ a^{c*}(\mathbf{p},-) \end{pmatrix}, \tag{18}$$

in the form

$$\Phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p})e^{ipx} + i\begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A^{c*}(\mathbf{p})e^{-ipx}\right]. \tag{19}$$

The Dirac field is then

$$\begin{aligned} \psi(x) &= \left(m - \gamma^a\partial_a\right)\beta\Phi(x) \\ &= \left(m - \gamma^a\partial_a\right)\beta\frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right] \\ &= \frac{1}{\sqrt{2}}\left[\chi_1(x) + i\chi_2(x)\right]. \end{aligned}$$

It satisfies the Dirac equation

$$\left(\gamma^a\partial_a + m\right)\psi(x) = 0 \tag{20}$$

because the Majorana fields $\chi_1$ and $\chi_2$ do.

We have defined Majorana and Dirac fields in terms of Weinberg's choice of Dirac matrices. If one uses a different set of Dirac matrices

$$\gamma^{a'} = S\gamma^a S^{-1}, \quad \beta' = S\beta S^{-1}, \tag{21}$$

then the fields and lawns should be multiplied from the left by the non-singular matrix $S$:

$$\Phi'(x) = S\Phi(x), \quad \psi'(x) = \psi(x), \quad etc. \tag{22}$$

Back to continuous groups $G$, Lie groups.

Say $g(\alpha) \in G$ is labeled by the parameters $\alpha_a$ for $a = 1 \ldots N$ with

$$g(0) = e \quad \text{the identity element}$$

Suppose $D(\alpha)$ is a representation as linear operators or matrices with

$$D(0) = 1.$$

Expand $D(\alpha)$ near 1

$$D(\epsilon) = 1 + i\epsilon_a X_a + \cdots$$

so that

$$X_a = -i \frac{\partial D(\alpha)}{\partial \alpha_a}\Big|_{\alpha=0}.$$

These are the generators of the Lie algebra.

The exponential parametrization of the group is

$$D(\alpha) = \lim_{k\to\infty} \left(1 + i\alpha_a X_a/k\right)^k = e^{i\alpha_a X_a}.$$

Note if

$$D(\lambda) = e^{i\lambda\alpha_a X_a} \quad \text{then}$$

$$D(\lambda_1)D(\lambda_2) = D(\lambda_1 + \lambda_2)$$

because $\alpha_a X_a$ commutes with itself. But in general

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a + \beta_a)X_a}$$

Instead

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

and one may show for small $\alpha, \beta$, that

$$i\delta_a X_a = i\alpha_a X_a + i\beta_a X_a - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + \cdots$$

So

$$[X_a, X_b] = i f_{abc} X_c$$

with

$$f_{abc} = -f_{bac}.$$

If the $X_a$'s are hermitian, then $e^{i\alpha_a X_a}$ for real $\alpha_a$ are unitary and

$$[X_a, X_b]^\dagger = -i f_{abc}^* X_c$$

$$= [X_b, X_a] = i f_{bac} X_c = -i f_{abc} X_c$$

So $f_{abc}^* = f_{abc}$ — the structure constants are real for unitary reps. of Lie groups.

Note that

$$i \delta_a X_a = i(\alpha_a + \beta_a) X_a - \tfrac{1}{2} i \alpha_a \beta_b f_{abc} X_c$$

In fact the f's determine the law of group multiplication at least near $D(0) = I$.

The generators $X_a$ satisfy the Jacobi identity

$$[X_a, [X_b, X_c]] + [X_b, [X_c, X_a]] + [X_c, [X_a, X_b]] = 0$$

The matrices $T_{ac}^b = i f_{abc}$ are the generators in the adjoint rep

$$[T^a, T^b] = i f_{abc} T^c$$

Since the f' are real, the $T^a$'s are purely imaginary,

$$T_a^* = -T_a.$$

The f's determine the multiplication law for their representation

$$D(\alpha)\, D(\beta) = D(\delta),$$

but since

$$D(g_1)\, D(g_2) = D(g_1 g_2),$$

the only real change when the f's are changed is in the parametrization $g_1 = g(\alpha)$, etc.

If $\qquad X'_a = L_{ab} X_b,$ then

$$[X'_a, X'_b] = L_{ad} L_{be} [X_d, X_e]$$

$$= i L_{ad} L_{be} f_{dec} X_c$$

$$= i L_{ad} L_{be} f_{deg} L^{-1}_{gh} L_{hc} X_c$$

$$= i L_{ad} L_{be} f_{deg} L^{-1}_{gh} X'_h$$

So

$$f'_{abc} = L_{ad} L_{be} f_{deg} L^{-1}_{gc} .$$

The new adjoint rep then is

$$(T'_a)_{bc} = L_{ad} L_{be} (T_d)_{eg} L^{-1}_{gc} \qquad \text{or}$$

$$T'_a = L_{ad} \; L T_d L^{-1}$$

Now

$$\mathrm{Tr}\, T^a T^b \to \mathrm{Tr}\, T_a' T_b' \quad \text{and}$$

$$\mathrm{Tr}(T_a' T_b') = L_{ac} L_{bd} \,\mathrm{Tr}(L T_c' L^{-1} L T_d' L^{-1})$$

$$= L_{ac} L_{bd} \,\mathrm{Tr}(T_c T_d)$$

Now the matrix

$$S_{cd} = \mathrm{Tr}(T_c T_d) = \mathrm{Tr}(T_d T_c) = S_{dc}$$

is real and symmetric. So

$$S'_{ab} = \mathrm{tr}(T_a' T_b') \quad \text{is}$$

$$S' = L S L^T$$

So we can make $S'$ diagonal by making $L$ orthogonal. So

$$\mathrm{Tr}\, T_a' T_b' = k^a \delta_{ab}.$$

We may rescale these with another $L$ (diagonal this time) to get

$$\mathrm{Tr}\, T_a T_b = \pm \delta_{ab},$$

If all the signs are $+$, then the group is compact. In this case

$$\mathrm{Tr}\, T_a T_b = \lambda\, \delta_{ab}$$

So now, dropping primes,

$$[T_a, T_b] = i f_{abc} T_c$$

$$\mathrm{tr}\,[T_a, T_b] T_c = i f_{abe}\, \mathrm{tr}\, T_e T_c = i\lambda f_{abc}$$

or

$$f_{abc} = -\frac{i}{\lambda}\, \mathrm{Tr}\left([T_a, T_b] T_c\right)$$

$$= -\frac{i}{\lambda}\, \mathrm{tr}\left(T_a T_b T_c - T_b T_a T_c\right)$$

$$= -\frac{i}{\lambda}\, \mathrm{tr}\left(T_b T_c T_a - T_c T_b T_a\right)$$

$$= -\frac{i}{\lambda}\, \mathrm{tr}\, [T_b, T_c] T_a = f_{bca}$$

So

$$f_{abc} = f_{bca} = f_{cab} = -f_{bac} = -f_{acb} = -f_{cba}$$

The f's are totally anti-symmetric.

$$(T^b)_{ac} = i f_{abc} = -i f_{cba} = -(T^b)_{ca}$$

So the T's are imaginary & anti-symmetric. They are Hermitian.

So if the group is compact, then the generators of the adjoint rep are hermitian, and the matrices of the adjoint rep. are unitary

$$D(\alpha) = \exp(i \alpha_a T_a)$$

$$D^\dagger(\alpha) = \exp(-i \alpha_a T_a) = D(\alpha)^{-1}.$$

The dimension of the adjoint rep is the number of generators — the range of a, b, c.

So compact groups have finite-dimensional unitary representations.

Noncompact groups — like $SL(2, \mathbb{C})$ & the Lorentz group — do not.

# Simple algebras and groups

An invariant subalgebra is a set of $X_a$'s such that for all $Y_b$ in the whole Lie algebra

$$[X_a, Y_b] = i f_{abc} X_c$$

That is $f_{abc} = 0$ for $a \in$ subalgebra unless $c \in$ sub algebra.

The whole algebra & 0 are trivially invariant subalgebras.

A group with no non-trivial invariant subalgebras is <u>simple</u>.

A group with no non-trivial abelian invariant subalgebras is semi-simple.

The adjoint rep of a simple compact Lie algebra is irreducible. We have

$$\mathrm{Tr}(T_a T_b) = \lambda \delta_{ab}.$$

If D is reducible, then $\exists$ an invariant subspace, spanned by $|r\rangle$ $r = 1 \cdots K$. Let $x = K+1, \cdots N$. Let's skip proof.

The simplest compact Lie algebra is $SU(2)$

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad J_i^\dagger = J_i$$

Let's diagonalize $J_3$. Assume a finite-dimensional rep. with $j$ the higest ev of $J_3$:

$$J_3 |j,\alpha\rangle = j|j,\alpha\rangle$$

where $\alpha$ labels degenerate e.v.'s taken to be on

$$\langle j,\alpha|j,\beta\rangle = \delta_{\alpha\beta}.$$

Let $$J^\pm = \frac{1}{\sqrt{2}}(J_1 \pm iJ_2),$$ elevator ops. (raising & lowering operators)

Then
$$[J_3, J^\pm] = \pm J^\pm$$ and
$$[J^+, J^-] = J_3 .$$

So if $$J_3 |m,\alpha\rangle = m|m,\alpha\rangle,$$ then

$$J_3 J^\pm |m,\alpha\rangle = (J^\pm J_3 \pm J^\pm)|m,\alpha\rangle$$
$$= (m\pm 1) J^\pm |m,\alpha\rangle .$$

$\nexists$ state with $J_3' = j+1$ so

$$J^+|j,\alpha\rangle = 0 \qquad \forall\alpha.$$

$J^- |j,\alpha\rangle$ has $J_3' = j-1$ so we may define

$$|j-1,\alpha\rangle = \frac{1}{N_j(\alpha)} J^- |j,\alpha\rangle, \qquad N_j \text{ a norm factor}$$

So

$$N_j(\beta)^* N_j(\alpha) \langle j-1, \beta | j-1, \alpha\rangle$$

$$= \langle j\beta | J^+ J^- | j\alpha\rangle = \langle j\beta | [J^+, J^-] | j\alpha\rangle$$

$$= \langle j\beta | J_3 | j\alpha\rangle = j \langle j\beta | j\alpha\rangle = j\, \delta_{\alpha\beta}.$$

Now set $N_j(\alpha) = \sqrt{j} = N_j$ so

$$\langle j-1\, \beta | j-1\, \alpha\rangle = \delta_{\alpha\beta}.$$

Also

$$J^+ | j-1, \alpha\rangle = \frac{1}{N_j} J^+ J^- | j\alpha\rangle = \frac{1}{N_j} [J^+, J^-] | j\alpha\rangle$$

$$= \frac{J_3}{N_j} | j\alpha\rangle = \frac{j}{N_j} | j\alpha\rangle = \sqrt{j}\, | j\alpha\rangle = N_j | j\alpha\rangle.$$

So $J^\pm$ raise and lower $J_3'$ but don't change $\alpha$.

$$J^- | j-1, \alpha\rangle = N_{j-1} | j-2, \alpha\rangle$$

$$J^+ | j-2, \alpha\rangle = N_{j-1} | j-1, \alpha\rangle.$$

There is a tower of states

$$J^- \,|j-k, \alpha\rangle = N_{j-k} \,|j-k-1, \alpha\rangle$$

$$J^+ \,|j-k-1, \alpha\rangle = N_{j-k} \,|j-k, \alpha\rangle.$$

Take the $N_j$'s real. Then

$$N_{j-k}^2 = N_{j-k}^2 \langle j-k-1, \alpha | j-k-1, \alpha\rangle$$

$$= \langle j-k, \alpha | J^+ J^- | j-k, \alpha\rangle$$

$$= \langle j-k, \alpha | [J^+, J^-] + J^- J^+ | j-k, \alpha\rangle$$

$$= \langle j-k, \alpha | J_3 | j-k, \alpha\rangle + N_{j-k+1}^2 \langle j-k+1, \alpha | j-k+1, \alpha\rangle$$

$$= j-k + N_{j-k+1}^2$$ a recursion relation

$$N_j^2 = j$$

$$N_{j-1}^2 - N_j^2 = j-1$$

...

$$N_{j-k+1}^2 - N_{j-k}^2 = j-k+1$$

$$N_{j-k}^2 - N_{j-k+1}^2 = j-k$$

---

$$N_{j-k}^2 = j + j-1 + \cdots \quad j-k+1 + j-k$$

$$= (k+1)j - \sum_{i=1}^{k} i = (k+1)j - \frac{k(k+1)}{2}$$

$$= \frac{1}{2}(k+1)(2j-k)$$

Let $k = j - m$ so that $|j-k,\alpha\rangle = |m,\alpha\rangle$. Then

$$N_m = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

We've assumed this rep. is finite dimensional. So

$$J^- |j-l,\alpha\rangle = 0$$

$$N_{j-l}^2 = \langle j-l\,\alpha | J^+ J^- | j-l\,\alpha\rangle = 0$$

$$N_{j-l}^2 = \frac{(2j-l)(l+1)}{2} = 0$$

So $\quad 2j = l \qquad j = \frac{l}{2}$.

So space breaks up into copies, one for each $\alpha$, each invariant under $\vec{J}$. But rep. is assumed to be irreducible. So $\alpha$ is unique. We drop it.

In fact, all the reps. of $[J_i, J_j] = i\epsilon_{ijk} J_k$ are of this form.

The usual notation is

$$|j,m\rangle = |m,\alpha\rangle = |j-k,\alpha\rangle.$$

$$\langle j', m' | J_3 | j, m \rangle = m \, \delta_{m'm} \, \delta_{j'j}$$

$$\langle j' m' | J^+ | j m \rangle = \sqrt{(j+m+1)(j-m)/2} \quad \delta_{m', m+1} \, \delta_{j'j}$$

$$\langle j' m' | J^- | j m \rangle = \sqrt{(j+m)(j-m+1)/2} \quad \delta_{m', m-1} \, \delta_{j'j}$$

The spin $j$ rep of $SU(2)$ is

$$[J_a^j]_{k\ell} = \langle j, j+1-k | J_a | j, j+1-\ell \rangle$$

where rows & columns go from $1$ to $2j+1$.
Usually we say

$$[J_a^j]_{m'm} = \langle j m' | J_a | j m \rangle$$

$m', m$ from $-j$ to $j$ in steps of $1$.

$$J_i^{1/2} = \frac{1}{2} \sigma_i$$

$$J_1^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$J_3^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Tensor products

$$|i,x\rangle = |i\rangle\,|x\rangle$$

$$D(g)\,|i,x\rangle = |j,y\rangle\ \left[D_{1\otimes 2}(g)\right]_{jy\,ix}$$

$$= |j\rangle\,|y\rangle\quad D_1(g)_{ji}\quad D_2(g)_{yx}$$

$$= \Big(|j\rangle\,D_1(g)_{ji}\Big)\Big(|y\rangle\,D_2(g)_{yx}\Big)$$

Since for small $\alpha_a$, $D(g) = \exp(i\alpha_a J_a) \simeq 1 + i\alpha_a J_a$,

$$(1 + i\alpha_a J_a)\,|i,x\rangle = |jy\rangle\langle jy|\,(1 + i\alpha_a J_a)\,|ix\rangle$$

$$= |j,y\rangle\left(\delta_{ij}\,\delta_{yx} + i\alpha_a\, J^{1\otimes 2}_{a\;jy\,ix}\right)$$

$$= |j,y\rangle\left(\delta_{ji} + i\alpha_a\, J^{1}_{a\,ji}\right)\left(\delta_{yx} + i\alpha_a\, J^{2}_{a\,yx}\right)$$

So
$$J^{1\otimes 2}_{a\;jy\,ix} = J^{1}_{a\,ji}\,\delta_{yx} + \delta_{ji}\,J^{2}_{a\,yx}\qquad\text{or}$$

$$J^{1\otimes 2}_a = J^1_a + J^2_a$$

$$J_a\big(|i\rangle\,|x\rangle\big) = \big(J_a|i\rangle\big)\,|x\rangle + |i\rangle\big(J_a|x\rangle\big)$$

Our basis has $J_3$ diagonal, so

$$J_3 \left( |j_1 m_1\rangle |j_2 m_2\rangle \right) = (m_1 + m_2) \left( |j_1 m_1\rangle |j_2 m_2\rangle \right)$$

E.g.,

$$|3/2, 3/2\rangle = |1/2, 1/2\rangle |1, 1\rangle$$

is the unique "highest-weight" state $J_3' = 3/2$. Now

$$J^- |3/2, 3/2\rangle = J^- \left( |1/2, 1/2\rangle |1, 1\rangle \right) = \sqrt{\tfrac{3}{2}}\ |3/2, 1/2\rangle$$

$$= J^{1/2\,-} |1/2, 1/2\rangle |1, 1\rangle + |1/2, 1/2\rangle J^{1\,-} |1, 1\rangle$$

$$= \sqrt{\tfrac{1}{2}}\ |1/2, -1/2\rangle |1, 1\rangle + |1/2, 1/2\rangle |1, 0\rangle \qquad \text{or}$$

$$|3/2, 1/2\rangle = \sqrt{\tfrac{1}{3}}\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle |1, 1\rangle + \sqrt{\tfrac{2}{3}}\ |1/2, 1/2\rangle |1, 0\rangle .$$

Similarly, $J^- |3/2, 1/2\rangle$ gives

$$|3/2, -1/2\rangle = \sqrt{\tfrac{2}{3}}\ |1/2, -1/2\rangle |1, 0\rangle + \sqrt{\tfrac{1}{3}}\ |1/2, 1/2\rangle |1, -1\rangle$$

$$|3/2, -3/2\rangle = |1/2, -1/2\rangle |1, -1\rangle .$$

The states $\perp$ to $|3/2, 1/2\rangle$ and $|3/2, -1/2\rangle$ are

$$|1/2, 1/2\rangle = \sqrt{\tfrac{2}{3}}\ |1/2, -\tfrac{1}{2}\rangle |1, 1\rangle - \sqrt{\tfrac{1}{3}}\ |1/2, 1/2\rangle |1, 0\rangle$$

$$|1/2, -1/2\rangle = \sqrt{\tfrac{1}{3}}\ |1/2, -1/2\rangle |1, 0\rangle - \sqrt{\tfrac{2}{3}}\ |1/2, 1/2\rangle |1, -1\rangle .$$

Since $J^- |1/2, 1/2\rangle = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2})/2}\ |1/2, -\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\ |1/2, -1/2\rangle$, The phase of $|1/2,-1/2\rangle$ relative to that of $|1/2,1/2\rangle$ is determined by $J^-$.

A tensor operator is a set of operators that transform like an irrep of the algebra.

A spin-s tensor operator $O^s_m$ for $m=-s,\cdots s$ transforms as

$$[J_a, O^s_m] = O^s_{m'} (J^s_a)_{m'm}.$$

Ex. $J_a = L_a = \epsilon_{abc} r_b p_c$

$$[J_a, r_b] = \epsilon_{acd} [r_c p_d, r_b]$$

$$= -i \epsilon_{acd} r_c \delta_{bd}$$

$$= -i \epsilon_{acb} r_c = r_c (J^{adj}_a)_{cb}$$

Since $$(J^{adj}_a)_{cb} = i \epsilon_{cab}.$$

$$J_a O^s_\ell |j m \alpha\rangle = [J_a, O^s_\ell] |j m \alpha\rangle + O^s_\ell J_a |j m \alpha\rangle$$

$$= O^s_{\ell'} |j m \alpha\rangle (J^s_a)_{\ell' \ell} + O^s_\ell |j, m', \alpha\rangle (J^j_a)_{m' m}$$

So $O^s_\ell |j m \alpha\rangle$ transforms as $s \otimes j$

For $a = 3$, in $J_3$-diagonal reps.,

$$J_3 O^s_\ell |j m \alpha\rangle = \ell\, O^s_\ell |j m \alpha\rangle + m\, O^s_\ell |j m \alpha\rangle$$

$$= (\ell + m) O^s_\ell |j m \alpha\rangle \qquad \text{i.e.}$$

$$J_3' = \ell + m.$$

One may show that

$$\sum_{\ell = M-j}^{M+j} O^s_\ell |j, M-\ell, \alpha\rangle \langle s, j; \ell\, M-\ell | J, M\rangle = k_j |J, M\rangle$$

and that

$$O^s_\ell |j m \alpha\rangle = \sum_{J=|j-s|}^{j+s} \langle J, \ell+m | s, j; \ell, m\rangle k_J |J, \ell+m\rangle$$

geometry $\Rightarrow$ Clebsch-Gordan coefficients

The physics leads to

$$k_J |J, \ell+m\rangle = \sum_\beta k_{\alpha\beta} |J, \ell+m, \beta\rangle$$

The $k_{\alpha\beta}$ depend on $\alpha, j, O^s$ and on $\beta$ and $J$.

But $k_{\alpha\beta}$ are independent of $l$ and $m$.

So we need to know $k_{\alpha\beta}$ only for one value of $l+m$. These reduced matrix elements are written as

$$k_{\alpha\beta} \equiv \langle J, \beta | O^s | j, \alpha \rangle.$$

Thus the Wigner-Eckart theorem:

$$\langle J, m', \beta | O^s_l | j, m\, \alpha \rangle = \delta_{m', l+m} \underbrace{\langle J, l+m | s, j; l, m \rangle}_{\text{C-G coeffs}}$$

$$\cdot \underbrace{\langle J, \beta | O^s | j, \alpha \rangle}_{\text{reduced physics}}$$

So if we know any non-zero matrix element of a tensor operator between states of same given $J, \beta$ and $j, \alpha$, we can compute all the others using the algebra.

Cartan's subalgebra is a maximal set of commuting hermitian generators $H_i$ $i=1\dots m$ in an irred. unitary rep $D$.

$$H_i = H_i^{+} \qquad [H_i, H_j] = 0.$$

$$\mathrm{tr}\,(H_i H_j) = k_D\, \delta_{ij} \qquad ij = 1\dots m$$

$m$ is rank of algebra.

$$H_i\, |\mu, x, D\rangle = \mu_i\, |\mu, x, D\rangle$$

The e.v.'s $\mu_i$ are the <u>weights</u>. They are real. $\vec{\mu} = (\mu_1, \cdots \mu_m)$ is the weight vector.

The <u>roots</u> are the weights of the adjoint rep.

Now in the adjoint representation, the index that labels the generators also labels the rows and columns of the generators, the states – the vectors – of the adjoint rep. correspond to generators. The vector associated with the generator $X_a$ we may call

$$|X_a\rangle \longleftrightarrow X_a$$

and also

$$\alpha|X_a\rangle + \beta|X_b\rangle = |\alpha X_a + \beta X_b\rangle \leftrightarrow \alpha X_a + \beta X_b$$

Use the scalar product

$$\langle X_a | X_b\rangle = \lambda^{-1}\, \mathrm{tr}\,(X_a^{+} X_b) \qquad , \lambda > 0$$

Now here's something amusing!

$$X_a|X_b\rangle = |X_c\rangle\langle X_c|X_a|X_b\rangle = |X_c\rangle (T_a)_{cb}$$

$$= i f_{cab}|X_c\rangle = i f_{abc}|X_c\rangle$$

$$= |i f_{abc} X_c\rangle = |[X_a, X_b]\rangle.$$

So if $H_i$ and $H_j$ are in the Cartan subalgebra, then

$$H_i|H_j\rangle = |[H_i, H_j]\rangle = 0.$$

So the states corresponding to the Cartan generators have zero weight vectors, $\vec{\mu} = 0$.

$$\langle H_i | H_j \rangle = \lambda^{-1} \operatorname{tr}(H_i^\dagger H_j) = \lambda^{-1} \operatorname{tr}(H_i H_j) = \delta_{ij}$$

is the normalization that's convenient

The other states of the adjoint rep $|E_\alpha\rangle$ have non-zero weight vectors $\vec{\alpha}$

$$H_i | E_\alpha \rangle = \alpha_i |E_\alpha\rangle = |\alpha_i E_\alpha\rangle$$

$$\| \quad |[H_i, E_\alpha]\rangle \qquad , \vec{\alpha} \text{ is real}$$

where $E_\alpha$ are the generators corresponding to the states $|E_\alpha\rangle$. So

$$[H_i, E_\alpha] = \alpha_i E_\alpha .$$

Take the adjoint

$$[E_\alpha^\dagger, H_i] = \alpha_i E_\alpha^\dagger$$

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

One sets $\quad E_{-\alpha} = E_\alpha^\dagger \qquad$ (like $J_- = J_+^\dagger$)

Normalization

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \operatorname{tr}(E_\alpha^\dagger E_\beta) = \delta_{\alpha\beta}$$

$$= \prod_{i=1}^{m} \delta_{\alpha_i \beta_i} ,$$

The weights $\alpha_i$ of the adjoint rep. are called <u>roots</u>, and $\vec{\alpha} = (\alpha_1, \cdots, \alpha_m)$ is a <u>root vector</u>

Just as $[J_3, J_\pm] = \pm J_\pm$ allows one to find the irreducible reps. of $SU(2)$ by using the elevator operators $J_\pm$, so too one may use $[H_i, E_{\pm\alpha}] = \pm\alpha_i E_{\pm\alpha}$ to find the irreps of any compact Lie group. The first step is to take an ev. $|\mu, D\rangle$ with weight $\vec{\mu}$ in rep $D$

$$H_i |\mu, D\rangle = \mu_i |\mu, D\rangle$$

and note that

$$H_i E_{\pm\alpha} |\mu, D\rangle = \left([H_i, E_{\pm\alpha}] + E_{\pm\alpha} H_i\right) |\mu, D\rangle$$

$$= \left(\pm\alpha_i E_{\pm\alpha} + E_{\pm\alpha} H_i\right) |\mu, D\rangle$$

$$= \left(\pm\alpha_i E_{\pm\alpha} + E_{\pm\alpha} \mu_i\right) |\mu, D\rangle$$

$$= \left(\mu_i \pm \alpha_i\right) E_{\pm\alpha} |\mu, D\rangle.$$

So the weights $\mu_i$ differ by the roots $\alpha_i$ just as the $m$'s differ by $\pm 1$, the roots of $SU(2)$.

One may show that

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H$$

which is the analog of $[J^+, J^-] = J_3$. For more stuff on Lie algebras, see Georgi's book, <u>Lie Algebras and Particle Physics.</u>

Suppose that $|e\rangle$ is an e.v. of the hamiltonian $H$

$$H|e\rangle = e|e\rangle. \qquad H^{+} = H$$

Suppose that $H$ is invariant under the unitary transformation $U$:

$$U^{+} H U = U^{-1} H U = H.$$

So $$HU = UH.$$

$U$ may represent a translation or a rotation or a "rotation" of the "internal coordinates." The state $U|e\rangle$ will also be an e.v. of $H$ with the same energy $e$:

$$H(U|e\rangle) = UH|e\rangle = Ue|e\rangle = e(U|e\rangle).$$

The transformations implemented by $U$ will form a group, $G$. If the group is compact, then the set of states

$$U(g)|e\rangle$$

will span a finite dimensional space of e.v.'s of $H$ with the same energy $e$. Call a basis in this space $|e,i\rangle$. Then

$$U(g)|e,j\rangle = \sum_i |e,i\rangle\langle e,i|U(g)|e,j\rangle.$$