A convenient choice of Dirac matrices, used by Weinberg, is

$$\gamma = -i \begin{pmatrix} \mathbf{0} & \sigma \\ -\sigma & \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \gamma^0 = -i\beta = -i \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}. \tag{1}$$

They satisfy the anti-commutation relations

$$[\gamma^a, \gamma^b]_+ = 2\eta^{ab}, \tag{2}$$

in which the flat space-time metric is

$$\eta^{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Under hermitian conjugation, they transform as $\gamma^\dagger = \gamma$ and $(\gamma^0)^\dagger = -\gamma^0$. For this choice of Dirac matrices, we may define Majorana and Dirac fields in terms of the scalar-like lawn

$$\phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0 (p^0 + m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p}) e^{ipx} + i \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A^*(\mathbf{p}) e^{-ipx} \right] \tag{4}$$

where I and σ_2 are the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5}$$

$A(\mathbf{p})$ and $A^*(\mathbf{p})$ are the 2-vectors

$$A(\mathbf{p}) = \begin{pmatrix} a(\mathbf{p}, +) \\ a(\mathbf{p}, -) \end{pmatrix} \quad \text{and} \quad A^*(\mathbf{p}) = \begin{pmatrix} a^\dagger(\mathbf{p}, +) \\ a^\dagger(\mathbf{p}, -) \end{pmatrix}, \tag{6}$$

$p^0 = \sqrt{m^2 + \mathbf{p})^2}$, and $\hbar = c = 1$. The lawn $\phi(x)$ describes a single spin-one-half particle that is its own anti-particle.

Since $m^2 + p^2 = m^2 + \mathbf{p}^2 - (p^0)^2 = 0$, the lawn $\phi(x)$ satisfies the Klein-Gordon equation

$$(m^2 + \partial_0^2 - \nabla^2)\phi(x) = (m^2 - \eta^{ab}\partial_a\partial_b)\phi(x) = 0. \tag{7}$$

The Majorana field $\chi(x)$ is obtained from derivatives of the lawn $\phi(x)$:

$$\chi(x) = (m - \gamma^a \partial_a)\beta\phi(x). \tag{8}$$

It automatically satisfies the Dirac equation:

$$\begin{aligned}
(\gamma^a \partial_a + m)\,\chi(x) &= (\gamma^a \partial_a + m)\,(m - \gamma^a \partial_a)\,\beta\phi(x) \\
&= \left(m^2 - \gamma^a \gamma^b \partial_a \partial_b\right)\beta\phi(x) \\
&= \left(m^2 - \tfrac{1}{2}[\gamma^a, \gamma^b]_+ \partial_a \partial_b\right)\beta\phi(x) \\
&= \left(m^2 - \eta^{ab}\partial_a \partial_b\right)\beta\phi(x) \\
&= \beta\left(m^2 - \eta^{ab}\partial_a \partial_b\right)\phi(x) = 0.
\end{aligned}$$

Suppose that there are two spin-one-half particles of the same mass m described by the two operators $a_1(\mathbf{p}, \sigma)$ and $a_2(\mathbf{p}, \sigma)$ which satisfy the anti-commutation relations

$$[a_i(\mathbf{p}, \sigma), a_j^\dagger(\mathbf{p}', \sigma')]_+ = \delta_{\sigma\sigma'}\delta^3(\mathbf{p} - \mathbf{p}'). \tag{9}$$

Then by following Eqs.(4–9) and defining two 2-vectors $A_i(\mathbf{p}, \sigma)$ as in (6), we may construct the two lawns

$$\phi_i(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0 + m)}} \left[\begin{pmatrix} I \\ I \end{pmatrix} A_i(\mathbf{p}) e^{ipx} + i \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A_i^*(\mathbf{p}) e^{-ipx}\right] \tag{10}$$

and from them the two Majorana fields

$$\chi_i(x) = (m - \gamma^a \partial_a)\beta\phi_i(x) \tag{11}$$

which satisfy the Dirac equation

$$(\gamma^a \partial_a + m)\,\chi_i(x) = 0. \tag{12}$$

But because the two lawns $\phi_i(x)$ are of the same mass, we may combine them into the complex lawn

$$\Phi(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right]. \tag{13}$$

From the complex operators

$$a(\mathbf{p}, \sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p}, \sigma) + i a_2(\mathbf{p}, \sigma)\right] \tag{14}$$

and

$$a^c(\mathbf{p},\sigma) = \frac{1}{\sqrt{2}}\left[a_1(\mathbf{p},\sigma) - ia_2(\mathbf{p},\sigma)\right], \tag{15}$$

we may form the complex 2-vectors

$$A(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) + iA_2(\mathbf{p})\right] = \begin{pmatrix} a(\mathbf{p},+) \\ a(\mathbf{p},-) \end{pmatrix} \tag{16}$$

and

$$A^c(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right] = \begin{pmatrix} a^c(\mathbf{p},+) \\ a^c(\mathbf{p},-) \end{pmatrix}. \tag{17}$$

The complex lawn involves $A(\mathbf{p})$ and

$$A^{c*}(\mathbf{p}) = \frac{1}{\sqrt{2}}\left[A_1(\mathbf{p}) - iA_2(\mathbf{p})\right]^* = \frac{1}{\sqrt{2}}\left[A_1^*(\mathbf{p}) + iA_2^*(\mathbf{p})\right] = \begin{pmatrix} a^{c*}(\mathbf{p},+) \\ a^{c*}(\mathbf{p},-) \end{pmatrix}, \tag{18}$$

in the form

$$\Phi(x) = \int \frac{d^3p}{2\sqrt{(2\pi)^3 p^0(p^0+m)}}\left[\begin{pmatrix} I \\ I \end{pmatrix} A(\mathbf{p})e^{ipx} + i\begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} A^{c*}(\mathbf{p})e^{-ipx}\right]. \tag{19}$$

The Dirac field is then

$$\begin{aligned} \psi(x) &= (m - \gamma^a\partial_a)\,\beta\Phi(x) \\ &= (m - \gamma^a\partial_a)\,\beta\frac{1}{\sqrt{2}}\left[\phi_1(x) + i\phi_2(x)\right] \\ &= \frac{1}{\sqrt{2}}\left[\chi_1(x) + i\chi_2(x)\right]. \end{aligned}$$

It satisfies the Dirac equation

$$(\gamma^a\partial_a + m)\,\psi(x) = 0 \tag{20}$$

because the Majorana fields χ_1 and χ_2 do.

We have defined Majorana and Dirac fields in terms of Weinberg's choice of Dirac matrices. If one uses a different set of Dirac matrices

$$\gamma^{a'} = S\gamma^a S^{-1}, \quad \beta' = S\beta S^{-1}, \tag{21}$$

then the fields and lawns should be multiplied from the left by the non-singular matrix S:

$$\Phi'(x) = S\Phi(x), \quad \psi'(x) = \psi(x), \quad etc. \tag{22}$$