

## 8

### Cell structure

#### 8.1 Organelles

A **prokaryotic** cell is a bag of salty water, DNA, RNA, proteins, sugars, and so forth between 0.1 and 5  $\mu\text{m}$  in diameter surrounded by a tough cell wall. Prokaryotic cells have no internal structure or transport mechanism. They rely upon diffusion to move their molecules around.

**Eukaryotic** cells have diameters between 10 and 100  $\mu\text{m}$ , and nerve cells can be as long as one meter. A eukaryotic cell has a **nucleus** that contains the cell's DNA, is about 5  $\mu\text{m}$  in diameter, and constitutes one-tenth of the volume of the cell. A plasma membrane surrounds the nucleus in a double layer that extends into the cytosol as the **endoplasmic reticulum**. Inside the nucleus, the **nucleolus** is a factory that makes ribosomes.

**RNA polymerase II** transcribes a nucleotide sequence from DNA into **pre-mRNA** inside the nucleus. A **spliceosome** (made of snRNAs and proteins) then excises **intron** segments and redundant **exons** from the pre-mRNA. An **exportin** protein then transports the resulting mRNA through a **nuclear pore complex** into the cytosol.

The endoplasmic reticulum and the **Golgi apparatus** use ribosomes and other molecular devices to make and decorate proteins and glycosaminoglycans (GAGs, long chains of disaccharides) which the cell then moves to the plasma membrane in membrane-enclosed sacs that bud from the endoplasmic reticulum and from the Golgi apparatus and then fuse with the membranes of other organelles or with the plasma membrane. Ribosomes in the cytosol also translate mRNAs into proteins.

A **mitochondrion** is an organelle with an outer membrane and an inner membrane on which **ATP synthase** uses **pyruvate** and **NADH** (both made from glucose in the cytosol) to turn ADP into ATP. A red blood cell has no nucleus nor any mitochondria, but a liver cell can have 2000 mito-

chondria. Most animals inherit their mitochondria only from their mothers; mechanisms destroy the mitochondria of sperm after fertilization (Zhou et al., 2016).

**Endosomes** are membrane sacs that enclose external material that the cell has just swallowed by **endocytosis**. Endosomes pass much of this material to **lysosomes** which use acids and enzymes to digest this material and surplus cytosolic macromolecules into elementary molecules that the cell reuses. Cells oxidize toxic molecules in **peroxisomes**. **Proteasomes** are organelles that use **proteases** to dissolve the peptide bonds of proteins, decomposing them into reusable amino acids.

Because eukaryotic cells have volumes that are between 8 and  $10^9$  times bigger than prokaryotic cells, they can't rely upon diffusion to move molecules and bags of molecules around. They use three families of protein filaments: **Intermediate filaments** give structure and flexibility (they can stretch and contract) to the **cytoskeleton**, are 10 nm in diameter, and are dimers of dimers. Oriented **microtubules** (made of dimers of the proteins  $\alpha$ -tubulin and  $\beta$ -tubulin) position the membrane-enclosed organelles, are 25 nm in diameter, and are the cell's Fedex. The motor proteins **dynein** and **kinesin** use ATP to drag membrane-enclosed sacs of molecules along microtubules toward their (+) and (-) ends. Oriented **actin filaments** are made of the globular protein G-actin and are 7 nm in diameter. They can rapidly become longer by polymerization or shorter by depolymerization, and so enable the cell to change its shape and to move.

## 8.2 Membranes

The plasma membrane of an animal cell and the membranes of the endoplasmic reticulum, the Golgi apparatus, the endosomes, and other membrane-enclosed organelles are lipid bilayers about 5-nm thick studded with proteins. The lipid constituents are mainly phospholipids, sterols, and glycolipids.

There are four main phospholipids in membranes. Three of them are neutral: phosphatidylethanolamine (PE), phosphatidylcholine (PC), and sphingolipids (SL) have polar but neutral head groups. Phosphatidylserine (PS), has a negatively charged head group. In a living cell, **flippases** use ATP to move PS and to a lesser degree PE to the cytosolic layer of the plasma membrane, while **floppases** use ATP to move PC and SL to the outer layer. **Scramblases** help these phospholipids move in the opposite directions (Clark, 2011).

The lipid interior of a cell membrane is a barrier to ions and to polar molecules bigger than water, but not to hydrophobic molecules such as  $O_2$ ,

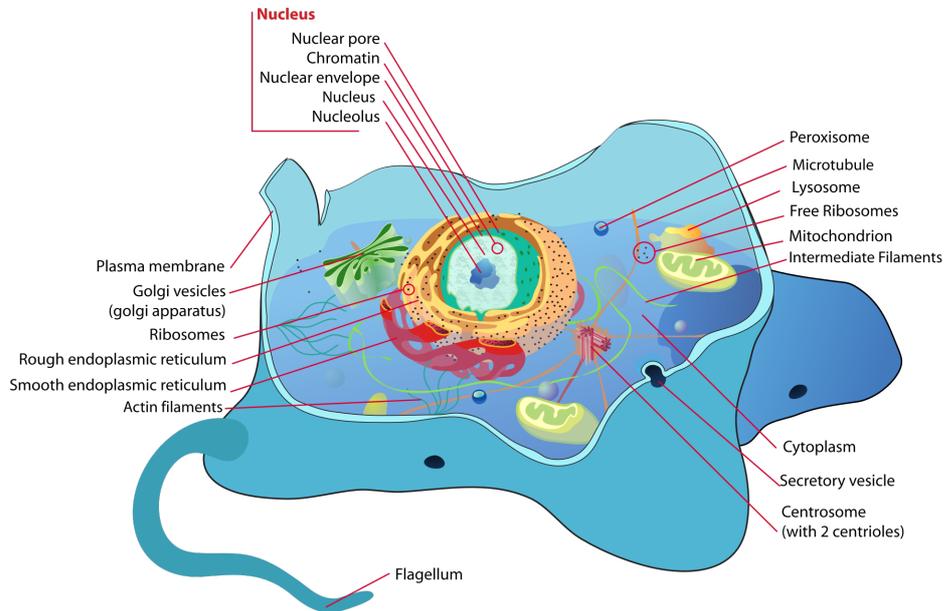


Figure 8.1 Cartoon of a eukaryotic cell. (Courtesy of Wikipedia.)

$\text{CO}_2$ ,  $\text{N}_2$ , and steroid hormones. **Transporters** and **channels**, the two classes of **membrane transport proteins**, control the passage of ions and big polar molecules across cell membranes. Transporters bind specific molecules and move them across the membrane. For instance,  **$\text{Na}^+ - \text{K}^+$  pumps** use ATP to drive 3  $\text{Na}^+$  ions out of a cell in each cycle while putting 2  $\text{K}^+$  ions into it. These **P-type transport ATPases** maintain the intracellular concentration of potassium 28 times higher than its extracellular concentration and that of sodium about 14 times lower than its extracellular concentration, as listed in table 8.1. In the cytosol, negative ions such as  $\text{Cl}^-$ ,  $\text{HCO}_3^-$ ,  $\text{PO}_4^{3-}$ , amino acids, proteins, nucleic acids, and metabolites carrying carboxyl or phosphate groups balance the positive charges of the 140 mM  $\text{K}^+$ .

**Water channels** or **aquaporins** pass  $10^9$  water molecules per second while blocking ions.  **$\text{K}^+$  leak channels** let potassium ions leak out of a cell. **Gated ion channels** open and close so as to allow specific ions  $\text{Na}^+$ ,  $\text{K}^+$ ,  $\text{Ca}^{2+}$ , or  $\text{Cl}^-$  to pass at up to  $10^8$  ions per second when needed. The  $\text{Na}^+ - \text{K}^+$  pumps and the  $\text{K}^+$  leak channels largely determine the concentra-

| Ion              | Intracellular concentration (mM)                     | Extracellular concentration (mM)                     |
|------------------|--|--|
| K <sup>+</sup>   | 140  | 5  |
| Na <sup>+</sup>  | 5-15   | 145  |
| Cl <sup>-</sup>  | 5-15   | 110  |
| Ca <sup>2+</sup> | 10 <sup>-4</sup>                                     | 1-2  |
| Mg <sup>2+</sup> | 0.5  | 1-2  |
| H <sup>+</sup>   | 7 × 10 <sup>-5</sup> = 10 <sup>-7.2</sup> M (pH 7.2) | 4 × 10 <sup>-5</sup> = 10 <sup>-7.4</sup> M (pH 7.4) |

Table 8.1 *Intracellular and extracellular concentrations (mM) of some ions in and near a typical mammalian cell (Alberts et al., 2008, chap. 11).*

tion gradients of Na<sup>+</sup> and K<sup>+</sup> and the 20–120 mV drop of the membrane potential across the plasma membrane.

### 8.3 Membrane potentials

**Gauss's law**  $\nabla \cdot \mathbf{D} = \rho$  says the divergence of the **electric displacement**  $\mathbf{D}$  is equal to the density  $\rho$  of **free charges** (as opposed to those of polar molecules). In electrostatic problems, Maxwell's equations reduce to Gauss's law and the static form  $\nabla \times \mathbf{E} = 0$  of Faraday's law which implies that the electric field  $\mathbf{E}$  is the gradient of an electrostatic potential  $\mathbf{E} = -\nabla V$ . (James Maxwell 1831–1879, Michael Faraday 1791–1867)

Across an interface with normal vector  $\hat{\mathbf{n}}$  between two dielectrics, the tangential electric field is continuous while the normal electric displacement jumps by the surface density of free charge  $\sigma$

$$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad \text{and} \quad \sigma = \hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1). \quad (8.1)$$

In a **linear isotropic dielectric**, the electric displacement  $\mathbf{D}$  is proportional to the electric field  $\mathbf{D} = \epsilon_m \mathbf{E}$ , where the **permittivity**  $\epsilon_m = K_m \epsilon_0$  of the material differs from that of the vacuum  $\epsilon_0$  by the **relative permittivity**  $K_m$ . The permittivity of the vacuum is the **electric constant**  $\epsilon_0 = 8.854 \times 10^{-12}$  F/m.

**Example 8.1** (Membrane potential). The plasma membrane is a phospholipid bilayer about 5 nm thick. If the potential drop across it is  $V_i - V_e = -50$

mV, then the electric field in the bilayer is  $-50 \text{ mV}/5 \text{ nm} = -10^7 \text{ V/m}$ . This very strong electric field points into the cell.  $\square$

**Example 8.2** (Charge density on and near inner leaflet). The field across the plasma membrane is mainly due to a negative surface charge density  $\sigma$  that lies on or near the cytosolic leaflet of the bilayer. If we apply Gauss's law (8.1) to a box whose top surface is in the middle of the bilayer, and whose bottom surface is at a depth in the cytosol where  $\mathbf{D} = 0$ , then we find that this charge density  $\sigma$  is equal to the electric displacement  $D$  on the top surface. In a lipid,  $D = \epsilon_m E \approx 2 \epsilon_0 E$ , so

$$\sigma \approx -2 \times 8.85 \times 10^{-12} \times 10^7 = -1.77 \times 10^{-4} \text{ C/m}^2. \quad (8.2)$$

The number of negative ions of charge  $-e$  per square meter that make up the surface charge density  $\sigma$  on or near the inner leaflet then is

$$n = \frac{\sigma}{-e} = \frac{-1.77 \times 10^{-4}}{-1.602 \times 10^{-19}} = 1.1 \times 10^{15} \quad (8.3)$$

or 1100 per square micron.

If we apply Gauss's law to the whole cell apart from the exterior leaflet of the plasma membrane, then we find that the net charge of a typical mammalian cell of radius  $10 \mu\text{m}$  is

$$Q = -1100 \cdot 4\pi \cdot 10^2 e = -1.4 \times 10^6 e = -2.2 \times 10^{-13} \text{ C} \quad (8.4)$$

or  $-0.22 \text{ pC}$ . If we were to assume that the net negative charge of a spherical cell were located in its nucleus, where most of the nucleotides are, then we would estimate the electric field at a distance  $r$  from the center of the cell as

$$E = \frac{Q}{4\pi K_c \epsilon_0 r^2} \quad (8.5)$$

in which the relative permittivity of the cytosol is  $K_c \approx 80$ . At  $r = 5 \mu\text{m}$ , the electric field would be

$$E = -\frac{2.2 \times 10^{-13}}{4\pi \cdot 80 \times 8.854 \times 10^{-12} \times 25 \times 10^{-12}} = -10^6 \text{ V/m}. \quad (8.6)$$

If the positive ions, the **cations**, screened the negative ions, the **anions**, then the net charge density would be uniform  $\rho = Q/(4\pi R^3/3)$  and inversely proportional to the radius  $R$  of the cell. The electric field then would be

$$E = \frac{Qr}{4\pi K_c \epsilon_0 R^3} \quad (8.7)$$

which at  $r = 5 \mu\text{m}$  is

$$E = - \frac{2.2 \times 10^{-13} \times 5 \times 10^{-6}}{4\pi 80 \times 8.854 \times 10^{-12} \times 10^3 \times 10^{-18}} = - 1.2 \times 10^5 \text{ V/m.} \quad (8.8)$$

In fact, however, the ions are much more effective at self-screening than a relative permittivity of 80 would suggest. Indeed, in the limit of high relative permittivity, as in a conducting metal, the net charge appears as a surface charge, and the electric field inside the cell is zero.

Incidentally, the cross-sectional area of a phospholipid is about 0.25 square nm. So there are some 4 million phospholipids in a square micron of each leaflet. Phosphatidylserine (PS) carries a negative charge of  $-e$ , lies only on the inner leaflet of a living cell, and makes up 4 percent of the plasma membrane of a liver cell and 7 percent of that of a red-blood cell. So PS makes up about 10 percent of the cytosolic leaflet, which means that the number of PS phospholipids per square micron is 400,000. Since 1100 would be enough to make the huge electric field in the membrane, it follows that positive ions of the cytosol cancel all but 0.25 percent of the charge of the PSs.  $\square$

#### 8.4 Potassium leak channels

The current  $\mathbf{J}$  due to a concentration  $c$  of an ion of charge  $q = ve$  in an electric field  $\mathbf{E}$  is given by the Nernst-Planck formula (7.58)

$$\mathbf{J} = D \left( -\nabla c + \frac{q\mathbf{E}c}{kT} \right). \quad (8.9)$$

Suppose the current is in a  $\text{K}^+$  leak channel that connects the cytosol to the outside of the cell. What voltage drop across the membrane would reduce the  $\text{K}^+$  current through the channel to zero? Setting  $\mathbf{J} = 0$ , and taking  $z$  to be the direction normal to the membrane, we find (7.58)

$$\frac{1}{c(z)} \frac{dc(z)}{dz} = \frac{qE_z}{kT} \quad (8.10)$$

which we can integrate to

$$\ln \frac{c_e}{c_i} = - \frac{ev}{kT} \int \frac{dV(z')}{dz'} dz' = \frac{ev(V_i - V_e)}{kT} \quad (8.11)$$

in which  $v$  is the valence of the ion of charge  $q = ve$ ,  $e > 0$  is the proton's charge,  $c_e$  is the extracellular concentration, and  $c_i$  is the intracellular

concentration. This voltage difference

$$V_i - V_e = \frac{kT}{e} \ln \frac{c_e}{c_i} \quad (8.12)$$

is the **Nernst potential**.

Using the equilibrium condition (8.11) and the  $K^+$  concentrations of table 8.1, we find that the voltage across the membrane needed to reduce the  $K^+$  current in a potassium leak channel to zero is

$$V_i - V_e = \frac{kT}{e} \ln \frac{c_e}{c_i} = \frac{1}{37} \ln \frac{c_e}{c_i} = \frac{1}{37} \ln \frac{5}{140} = -0.09 \text{ V} \quad (8.13)$$

or  $-90$  mV at 37 Celsius. The  $K^+$  concentrations of a typical neuron listed in table 8.2 give  $-104$  mV as the required membrane potential to stop the  $K^+$  flow through a potassium leak channel of a neuron at body temperature. Since the resting membrane potentials usually are less than  $-90$  mV,  $K^+$  ions do flow through the potassium leak channels, and the  $Na^+ - K^+$  pumps must keep on pumping.

### 8.5 The Debye-Hückel equation

If there are several kinds of ions of charge  $q_i$  and a fixed charge distribution  $\rho_f$ , then the Poisson-Boltzmann equation (7.67) is

$$-\epsilon_m \nabla^2 V = \rho_f + \sum_i \rho_{i,0} e^{-q_i V/kT}. \quad (8.14)$$

When the potential low enough and the temperature is high enough that  $qV/kT \ll 1$ , one can set

$$e^{-q_i V/kT} \approx 1 - \frac{q_i}{kT} V \quad (8.15)$$

and get the **linearized Poisson-Boltzmann** equation

$$-\epsilon_m \nabla^2 V = \rho_f + \sum_i \rho_{i,0} \left(1 - \frac{q_i}{kT} V\right) \quad (8.16)$$

also known as the **Debye-Hückel** equation.

Often, the free-charge distribution  $\rho_f$  occurs on the boundary of the region of interest as a boundary condition. The charge densities  $\rho_{i,0}$  often are the bulk densities which usually add to zero because cells are electrically neutral or nearly neutral. With these simplifications, the Debye-Hückel equation is

$$\nabla^2 V = \left( \sum_i \frac{\rho_{i,0} q_i}{\epsilon_m kT} \right) V. \quad (8.17)$$

The term inside the parentheses is the inverse of the square of the **Debye length**

$$\lambda_D = \left( \frac{\epsilon_m kT}{\sum_i \rho_{i,0} q_i} \right)^{1/2}. \quad (8.18)$$

In the cytosol of a typical mammalian cell, the concentration of positive ions, mainly  $K^+$ , is 150 mM or  $0.15N_A = 9.033 \times 10^{22}$  molecules per liter, which is  $9.033 \times 10^{25}$  per cubic meter. The concentration of negative ions is very similar. At 37 °C, the permittivity of the cytosol is  $\epsilon_c = 74.16 \epsilon_0 = 6.57 \times 10^{-10}$  F/m and  $kT/e = 1/37.4$  V. So the Debye length of the cytosol for monovalent ions  $q_i = \pm e$  is

$$\lambda_D = \left( \frac{6.57 \times 10^{-10}}{2(9.033) \times 10^{25} (37.4) 1.602 \times 10^{-19}} \right)^{1/2} = 7.79 \times 10^{-10} \text{ m} \quad (8.19)$$

or about 0.8 nm.

The **Bjerrum length**

$$\ell_B = \frac{e^2}{4\pi\epsilon_c kT} \quad (8.20)$$

has a similar value at 37 °C

$$\ell_B = \frac{1.602 \times 10^{-19} \times 37.4}{4\pi \times 74.16 \times 8.854 \times 10^{-12}} = 7.26 \times 10^{-10} \text{ m} \quad (8.21)$$

or 0.726 nm.

In one space dimension, the Debye-Hückel equation is

$$\frac{d^2V}{dz^2} = \frac{1}{\lambda_D^2} V. \quad (8.22)$$

**Example 8.3** (Field just inside a plasma membrane). We can use the one-dimensional Debye-Hückel equation (8.22) to estimate the electric field due to the negative surface charge  $\sigma$  of the phosphatidylserine on the inner leaflet of a cell membrane. Integrating twice, we find

$$V(z) = V_0 e^{-z/\lambda_D} \quad (8.23)$$

in which  $z > 0$  is the *distance* down into the cell, and  $V_0$  is a constant we will determine from the boundary condition due to the surface charge  $\sigma$ . This boundary condition is

$$E = -\frac{dV}{dz} = \frac{\sigma}{\epsilon_c} \quad (8.24)$$

in which  $\epsilon_c \approx 74 \epsilon_0$  is the permittivity of the cytosol. (If the cytosol and the extracellular fluid were solid dielectrics with no mobile ions, then the

boundary condition would be  $E = \sigma/(\epsilon_c + \epsilon_w)$  in which  $\epsilon_w$  is the permittivity of the extracellular fluid.) So we have

$$-\left. \frac{dV}{dz} \right|_{z=0} = \frac{V_0}{\lambda_D} = \frac{\sigma}{\epsilon_c} \quad (8.25)$$

or

$$V_0 = \frac{\lambda_D \sigma}{\epsilon_c}. \quad (8.26)$$

Thus the potential in the cytosol due to the PS on the inner leaflet is

$$V(z) = \frac{\lambda_D \sigma}{\epsilon_c} e^{-z/\lambda_D}. \quad (8.27)$$

The electric field in the cytosol falls off exponentially with the depth in units of the Debye length,  $z/\lambda_D$ . The huge fields we imagined in the thought experiments of example 8.2 do not exist except within about a nanometer of the inner leaflet of the membrane, within a **double layer** formed by the negative surface charge  $\sigma$  of the PS's and the counter ions, mainly  $K^+$ 's.

The charge density of positive ions is

$$\sigma_+ = -\epsilon_c \frac{d^2 V}{dz^2} = -\frac{\sigma}{\lambda_D} e^{-z/\lambda_D}, \quad (8.28)$$

and its integral is

$$-\int_0^\infty \frac{\sigma}{2\lambda_D} e^{-z/\lambda_D} dz = -\sigma \quad (8.29)$$

equal and opposite to the charge density of the PSs. The cell is neutral.  $\square$

The homogeneous Debye-Hückel equation in three spatial dimensions with spherical symmetry is

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = \frac{1}{\lambda_D^2} V \quad (8.30)$$

or more simply

$$\frac{d^2 (rV)}{dr^2} = \frac{rV}{\lambda_D^2}. \quad (8.31)$$

**Example 8.4** (Electric field about nucleus). The DNA of a cell and much of its RNA is negatively charged and in the nucleus. How big is the resulting electric field? The  $K^+$  ions and other positive ions hover about  $PO_4^-$  groups of the DNA and RNA screening most of their charges. For simplicity, however, let us imagine that the DNA lies within a sphere of radius  $R$  and charge  $Q$ , and that the  $K^+$  and other ions lie outside this sphere. The

resulting spherical symmetry let us use the Debye-Hueckel equation (8.31) whose solution is

$$r V(r) = v e^{-r/\lambda_D}. \quad (8.32)$$

From the boundary condition

$$E(R) = - \left. \frac{dV(r)}{dr} \right|_{r=R} = \frac{Q}{4\pi\epsilon_c R^2}, \quad (8.33)$$

we determine the constant  $v$  to be

$$v = \frac{Q}{4\pi\epsilon_c} \frac{\lambda_D}{\lambda_D + R} e^{R/\lambda_D}. \quad (8.34)$$

So the potential is

$$V(r) = \frac{Q}{4\pi\epsilon_c} \frac{\lambda_D}{\lambda_D + R} \frac{e^{-(r-R)/\lambda_D}}{r}. \quad (8.35)$$

The electric field at  $r > R$  then is

$$E(r) = \frac{Q}{4\pi\epsilon_c} \frac{\lambda_D + r}{\lambda_D + R} \frac{e^{-(r-R)/\lambda_D}}{r^2}. \quad (8.36)$$

How big is this electric field? The human genome contains  $3.2 \times 10^9$  base pairs. So its charge is  $-6.4 \times 10^9 e = -10^{-9}$  C or  $-1$  nC. So at the edge of a nucleus, at  $R = 4\mu\text{m}$ , the electric field (8.36) is  $5.6 \times 10^{11}$  V/m which is enough to ionize atoms. This is a gross overestimate because we ignored the  $\text{K}^+$  counterions inside the nucleus. Our formula (8.36) becomes much more reasonable for  $r > R$ . Since the Debye length (8.19) in the cytosol at  $37^\circ\text{C}$  is  $\lambda_D = 0.8$  nm, at  $r = 4.02\mu\text{m}$ , for instance, the exponential factor in the electric field (8.36) is

$$e^{-(r-R)/\lambda_D} = e^{-2 \times 10^{-8}/(8 \times 10^{-10})} = 10^{-11}, \quad (8.37)$$

and so the field is negligible at more than 20 nm from the nucleus.  $\square$

## 8.6 The Poisson-Boltzmann equation in one dimension

If there are several kinds of ions of charge  $q_i$  and no fixed charge distribution, then the Poisson-Boltzmann equation (7.67) is

$$-\epsilon \nabla^2 V = \sum_i \rho_{i,0} e^{-q_i V/kT}. \quad (8.38)$$

Gouy and Chapman solved this equation (Gouy, 1910; Chapman, 1913) in one dimension

$$-\epsilon \frac{d^2 V}{dz^2} = \sum_i \rho_{i,0} e^{-q_i V/kT}. \quad (8.39)$$

Let's consider a fluid that contains ions and counterions of charge  $\pm e$  whose charge densities in the bulk are equal,  $\rho_{+,0} = -\rho_{-,0} = \rho_0$ . Then the one-dimensional Gouy-Chapman equation (8.39) for monovalent ions is

$$-\epsilon \frac{d^2 V}{dz^2} = \rho_0 \left( e^{-eV/kT} - e^{eV/kT} \right) \quad (8.40)$$

in which  $z > 0$  is the distance away from the surface charge density  $\sigma$ . One may verify (exercise 8.1) that the second derivative of the potential

$$V(z) = \frac{2kT}{e} \ln \left[ \frac{1 + e^{-(z+z_0)/\lambda_D}}{1 - e^{-(z+z_0)/\lambda_D}} \right] \quad (8.41)$$

multiplied by  $-\epsilon$  is

$$-\epsilon \frac{d^2 V}{dz^2} = -\frac{4\epsilon kT}{e\lambda_D^2} \frac{e^{-(z+z_0)/\lambda_D} + e^{-3(z+z_0)/\lambda_D}}{(1 - e^{-2(z+z_0)/\lambda_D})^2}, \quad (8.42)$$

and that it is equal to the other side of the Gouy-Chapman equation (8.40) because of the definition (8.18) of the Debye length (exercise 8.2). One may adjust the parameter  $z_0$  to satisfy the boundary condition (8.25)

$$e^{-z_0/\lambda_D} = \frac{2kT\epsilon_c}{e\lambda_D\sigma} \left[ \sqrt{1 + \left( \frac{e\lambda_D\sigma}{2kT\epsilon_c} \right)^2} - 1 \right] \quad (8.43)$$

(exercise 8.3). The solution then at  $z \gg \lambda_D$  approaches the solution (8.27) to the Debye-Hückel equation (8.22) (exercise 8.4).

## 8.7 Neurons

Transporters, mainly sodium-potassium pumps, water channels, and gated ion channels, and channels, mainly the potassium leak channel, control the osmotic pressure of a cell and maintain the concentrations of sodium, potassium, and chloride at about 14, 140, and 14 mM inside the cell while the extracellular fluid has  $\text{Na}^+$ ,  $\text{K}^+$ , and  $\text{Cl}^-$  concentrations of 140, 3, and 146 mM as listed in table 8.2. The resulting resting voltage across the membrane about  $V_0 = -70$  mV, and the electric field points into the neuron.

The voltage drop across the membrane  $\Delta V = V_c - V_e$  is the same for all

| Ion       | Intracellular concentration (mM) | Extracellular concentration (mM) | Relative chord conductance $g_s/g_0$ |
|-----------|----------------------------------|----------------------------------|--------------------------------------|
| $K^+$     | 140                              | 3                                | 0.649                                |
| $Na^+$    | 14                               | 140                              | 0.026                                |
| $Cl^-$    | 14                               | 146                              | 0.325                                |
| $Ca^{2+}$ | $10^{-4}$                        | 1.5                              | 0                                    |

Table 8.2 *Intracellular and extracellular concentrations (mM) of some ions in and near a typical neuron (Hammond, 2015, chap. 3).*

three species  $Na^+$ ,  $K^+$ , and  $Cl^-$ . For each kind of ion, this voltage is the sum of the  $IR$  voltage and the Nernst potential (8.12)

$$\begin{aligned}\Delta V = V_c - V_e &= I_{Na^+} R_{Na^+} + V_{N,Na^+} = I_{K^+} R_{K^+} + V_{N,K^+} \\ &= I_{Cl^-} R_{Cl^-} + V_{N,Cl^-}\end{aligned}\quad (8.44)$$

where the Nernst potential for each species is

$$V_{N,s} = V_{N,s,c} - V_{N,s,e} = -\frac{kT}{z_s e} \ln \frac{c_{s,c}}{c_{s,e}} \quad (8.45)$$

in which  $z_s$  is the valence, which is unity for the principal ions  $Na^+$ ,  $K^+$ , and  $Cl^-$ . While the neuron is quiet with no signal running along the axon, the total current through the membrane vanishes. And so if  $g_s$  is the conductance of species  $s$ , then the sum of the currents  $j_s = (V_0 - V_{N,s}) g_s$  over the species is zero  $Na^+$ ,  $K^+$ , and  $Cl^-$

$$0 = \sum_s (V_0 - V_{N,s}) g_s. \quad (8.46)$$

The total conductance is

$$g_0 = \sum_s g_s. \quad (8.47)$$

In terms of it the resting voltage  $V_0$  is

$$V_0 = \sum_s \frac{g_s}{g_0} V_{N,s} \quad (8.48)$$

which is the **chord conductance formula**.

#### Further reading

The book *Molecular Biology of the Cell* may be the best textbook ever written. Chapters 22 and 23 of the excellent book *Molecular Driving Forces* by Ken Dill and Sarina Bromberg discuss the Poisson-Boltzmann and Debye-Hückel equations. Philip Nelson's fine book *Biological Physics* discusses the Gouy-Chapman equation.

### Exercises

- 8.1 Show that the second derivative of the Gouy-Chapman potential (8.41) is given by (8.42).
- 8.2 Show that the second derivative (8.42) of the Gouy-Chapman potential (8.41) is equal to the other side of the Gouy-Chapman equation (8.40).
- 8.3 Show that the formula (8.43) for  $z_0$  satisfies the boundary condition (8.25).
- 8.4 Take the limit  $z \rightarrow \infty$  of the Gouy-Chapman solution (8.41) and show that it is the solution (8.27) to the Debye-Hückel equation (8.22).