to the series for \( f(x) \)? The complex conjugate of the Fourier series (2.2) is

\[
\mathcal{F}^* = \sum_{n=-\infty}^{\infty} f_n^* \frac{e^{-inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} f_{-n}^* \frac{e^{inx}}{\sqrt{2\pi}}
\]

so the coefficients \( f_n(f^*) \) for \( f^*(x) \) are related to those \( f_n(f) \) for \( f(x) \) by

\[
f_n(f) = f_n^*(f) = f_{-n}^*(f).
\]

Thus if the function \( f(x) \) is real, then

\[
f_n(f) = f_n(f^*) = f_{-n}^*(f).
\]

Thus the Fourier coefficients \( f_n \) for a real function \( f(x) \) satisfy

\[
f_n = f_{-n}^*.
\]

**Example 2.1** (Fourier Series by Inspection). The doubly exponential function \( \exp(\exp(ix)) \) has the Fourier series

\[
\exp(\exp(ix)) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{inx}
\]

in which \( n! = n(n-1)\ldots1 \) is \( n \)-factorial with \( 0! \equiv 1 \).

**Example 2.2** (Beats). The sum of two sines \( f(x) = \sin\omega_1 x + \sin\omega_2 x \) of similar frequencies \( \omega_1 \approx \omega_2 \) is the product (exercise 2.1)

\[
f(x) = 2 \cos \frac{1}{2}(\omega_1 - \omega_2)x \sin \frac{1}{2}(\omega_1 + \omega_2)x
\]

in which the first factor \( \cos \frac{1}{2}(\omega_1 - \omega_2)x \) is the *beat* which modulates the second factor \( \sin \frac{1}{2}(\omega_1 + \omega_2)x \) as illustrated by Fig. 2.1.

**2.2 The Interval**

In equations (2.1–2.3), we singled out the interval \([0, 2\pi]\), but to represent a periodic function with period \(2\pi\), we could have used any interval of length \(2\pi\), such as the interval \([-\pi, \pi]\) or \([r, r + 2\pi]\)

\[
f_n = \int_{r}^{r+2\pi} e^{-inx} f(x) \frac{dx}{\sqrt{2\pi}}.
\]

This integral is independent of its lower limit \( r \) as long as the function \( f(x) \) is periodic with period \(2\pi\). The choice \( r = -\pi \) often is convenient. With this
2.3 Where to put the 2pi’s

Figure 2.1 The curve $\sin \omega_1 x + \sin \omega_2 x$ for $\omega_1 = 30$ and $\omega_2 = 32$.

choice of interval, the coefficient $f_n$ is the integral (2.3) shifted by $-\pi$

$$f_n = \int_{-\pi}^{\pi} e^{-inx} f(x) \frac{dx}{\sqrt{2\pi}}. \quad (2.11)$$

But if the function $f(x)$ is not periodic with period $2\pi$, then the Fourier coefficients (2.10) do depend upon $r$.

2.3 Where to put the 2pi’s

In Eqs. (2.2 & 2.3), we used the orthonormal functions $\exp(inx)/\sqrt{2\pi}$, and so we had factors of $1/\sqrt{2\pi}$ in both equations. If one gets tired of having so many explicit square roots, then one may set $d_n = f_n/\sqrt{2\pi}$ and write (2.2) and (2.3) as

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi} \int_{0}^{2\pi} dx e^{-inx} f(x). \quad (2.12)$$