to the series for $f(x)$? The complex conjugate of the Fourier series (2.2) is

$$f^*(x) = \sum_{n=-\infty}^{\infty} f_n^* \frac{e^{-inx}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} f_{-n}^* \frac{e^{inx}}{\sqrt{2\pi}} \tag{2.4}$$

so the coefficients $f_n(f^*)$ for $f^*(x)$ are related to those $f_n(f)$ for $f(x)$ by

$$f_n(f^*) = f_{-n}^*(f). \tag{2.5}$$

Thus if the function $f(x)$ is real, then

$$f_n(f) = f_n(f^*) = f_{-n}^*(f). \tag{2.6}$$

Thus the Fourier coefficients f_n for a real function $f(x)$ satisfy

$$f_n = f_{-n}^*. \tag{2.7}$$

Example 2.1 (Fourier Series by Inspection)**.** The doubly exponential function $\exp(\exp(ix))$ has the Fourier series

$$\exp\left(e^{ix}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\, e^{inx} \tag{2.8}$$

in which $n! = n(n-1)\ldots 1$ is n-factorial with $0! \equiv 1$. □

Example 2.2 (Beats)**.** The sum of two sines $f(x) = \sin\omega_1 x + \sin\omega_2 x$ of similar frequencies $\omega_1 \approx \omega_2$ is the product (exercise 2.1)

$$f(x) = 2\cos\tfrac{1}{2}(\omega_1 - \omega_2)x \;\; \sin\tfrac{1}{2}(\omega_1 + \omega_2)x \tag{2.9}$$

in which the first factor $\cos\frac{1}{2}(\omega_1 - \omega_2)x$ is the *beat* which modulates the second factor $\sin\frac{1}{2}(\omega_1 + \omega_2)x$ as illustrated by Fig. 2.1. □

2.2 The Interval

In equations (2.1–2.3), we singled out the interval $[0, 2\pi]$, but to represent a periodic function with period 2π, we could have used any interval of length 2π, such as the interval $[-\pi, \pi]$ or $[r, r + 2\pi]$

$$f_n = \int_r^{r+2\pi} e^{-inx}\, f(x)\, \frac{dx}{\sqrt{2\pi}}. \tag{2.10}$$

This integral is independent of its lower limit r as long as the function $f(x)$ is periodic with period 2π. The choice $r = -\pi$ often is convenient. With this

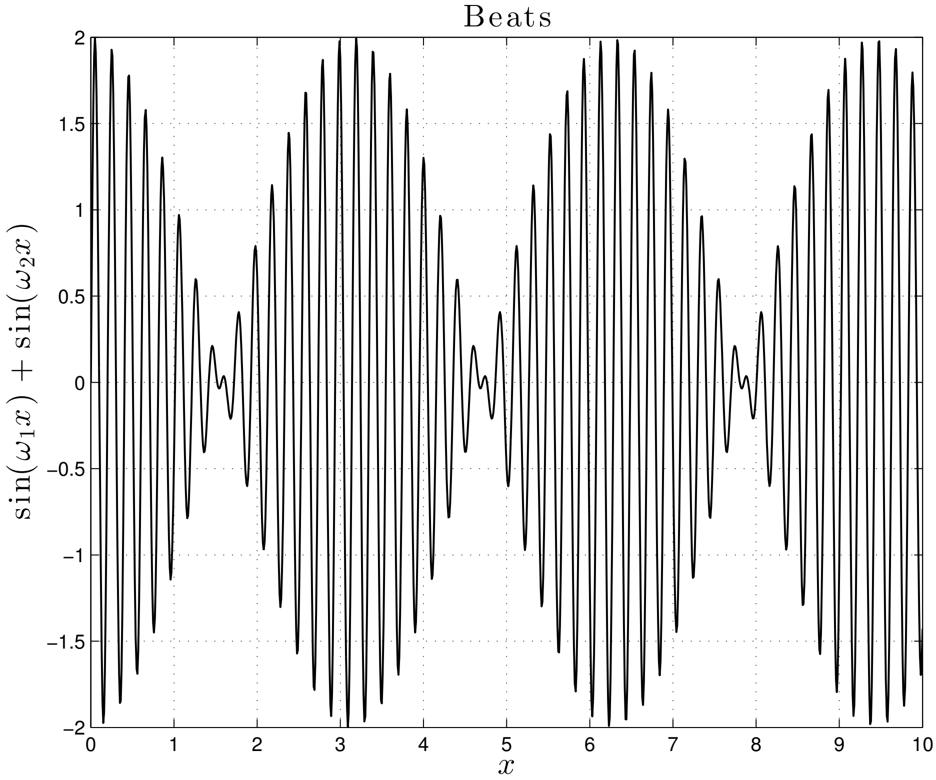

Figure 2.1 The curve $\sin \omega_1 x + \sin \omega_2 x$ for $\omega_1 = 30$ and $\omega_2 = 32$.

choice of interval, the coefficient f_n is the integral (2.3) shifted by $-\pi$

$$f_n = \int_{-\pi}^{\pi} e^{-inx}\, f(x)\, \frac{dx}{\sqrt{2\pi}}. \tag{2.11}$$

But if the function $f(x)$ is not periodic with period 2π, then the Fourier coefficients (2.10) do depend upon r.

2.3 Where to put the 2pi's

In Eqs.(2.2 & 2.3), we used the orthonormal functions $\exp(inx)/\sqrt{2\pi}$, and so we had factors of $1/\sqrt{2\pi}$ in both equations. If one gets tired of having so many explicit square roots, then one may set $d_n = f_n/\sqrt{2\pi}$ and write (2.2) and (2.3) as

$$f(x) = \sum_{n=-\infty}^{\infty} d_n\, e^{inx} \quad \text{and} \quad d_n = \frac{1}{2\pi} \int_0^{2\pi} dx\, e^{-inx}\, f(x). \tag{2.12}$$