the electric displacement $\boldsymbol{D} = \epsilon_m \boldsymbol{E}$ is proportional to the electric field $\boldsymbol{E}$ in which the **permittivity** $\epsilon_m = K_m \epsilon_0$ of the dielectric is a multiple K_m of the **electric constant** $\epsilon_0 = 8.85418782 \times 10^{-12}$ F/m.

If the fluid is in equilibrium, then the charge density ρ should be proportional to the Boltzmann factor

$$\rho = \rho_0 \, e^{-qV/kT}. \tag{7.63}$$

If the fluid is an linear isotropic dielectric with a constant permittivity ϵ_m, then Gauss's law (7.62) is

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \epsilon_m \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho. \tag{7.64}$$

Equilibrium implies time independence, and so the electric field is related to the gradient of the scalar potential V

$$\boldsymbol{E} = -\boldsymbol{\nabla} V. \tag{7.65}$$

Putting these equations (7.63–7.65) together, we arrive at the **Poisson-Boltzmann equation**

$$-\epsilon_m \, \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \, V = \rho_0 \, e^{-qV/kT}. \tag{7.66}$$

A more general form of the Poisson-Boltzmann equation applies when there are several kinds of ions of charge q_i and a fixed charge distribution ρ_f

$$-\epsilon_m \, \boldsymbol{\nabla}^2 \, V = \rho_f + \sum_i \rho_{i,0} \, e^{-q_i V/kT}. \tag{7.67}$$

This nonlinear partial differential equation is hard to solve analytically. Gouy and Chapman, however, found an exact solution for problems in one dimension (Gouy, 1910; Chapman, 1913), which we'll use in section 8.6. (Louis Gouy 1854-1926, David Chapman 1869-1958)

7.5 Reynolds number

Engineers mean two very different things by *stress* and *strain* which have similar meanings in everyday English. Consider a fluid between two large parallel plates of area A separated by a distance L in the vertical z-direction. Suppose a force F moves the top plate slowly in the x direction at a constant speed v, while the bottom plate remains stationary. The **shear stress** τ is the force applied to the top plate divided by its area,

$$\tau = \frac{F}{A} \qquad \text{shear stress.} \tag{7.68}$$

If the speed v is slow enough and the separation z big enough, then the liquid adjacent to and adsorbed by the upper plate moves at the speed of the upper plate, and the liquid adjacent to and adsorbed by the lower plate is stationary like the lower plate. The **shear strain** of the fluid at height z is the derivative of the displacement Δx in the x-direction with respect to the height

$$\gamma = \frac{d\Delta x}{dz} \qquad \text{shear strain.} \tag{7.69}$$

The displacement Δx is proportional to the time t, and its time derivative is the speed at height z

$$v(z) = \frac{d\Delta x}{dt}. \tag{7.70}$$

The **shear rate** is the z-derivative of $v(z)$. In **newtonian liquids**, the shear stress is proportional to the shear rate

$$\tau = \frac{F}{A} = \eta \, \frac{dv(z)}{dz}, \tag{7.71}$$

and the constant of proportionality is the **viscosity** η.

The **viscosity** η of water runs from 0.001003 kg/m·s at 20° C to 0.000692 kg/m·s at 37° C; that of corn syrup is $\eta = 1.3806$ kg/m·s at 25°C. (One kg/m·s = 1 Pa·s.) Stokes related the viscosity η of a fluid of particles of radius r to the viscous-friction coefficient ζ of the fluid

$$\eta = \frac{\zeta}{6\pi r}. \tag{7.72}$$

The dimensionless ratio of the mass ρAL times v^2/L, which has the same dimensions as acceleration, to the force F

$$\mathcal{R} = \frac{\rho A L v^2/L}{F} = \frac{\rho A L v^2/L}{\eta A v/L} = \frac{\rho L v}{\eta} \tag{7.73}$$

is the **Reynolds number** of a process. In numerous experiments, Reynolds showed that a process that has a Reynolds number $\mathcal{R} \gtrsim 10^3$ is turbulent, while one with a small Reynolds number $\mathcal{R} \ll 10^3$ flows smoothly in layers, a process called **laminar flow**. (Osborne Reynolds 1842–1912)

A pressure p forces a newtonian fluid of mass density ρ and viscosity η through a cylindrical pipe of radius R and length L (in the absence of gravity). The speed $v(r)$ is fastest in the center of the pipe, $r = 0$, and vanishes at the inner surface of the pipe, $r = R$. In laminar flow, the fluid does not accelerate, so the force $p2\pi r dr$ applied by the pressure to an annulus of width dr must equal the frictional force or drag due to the viscosity η.

That drag is the difference between the frictional force forward due to the faster fluid at r minus the frictional force forward due to the slower fluid at $r + dr$. Using our definition (7.71) of the viscosity (with $z \to r$), we find that the drag is $2\pi\eta L \, d(rdv(r)/dr)$. Equating the two forces and dividing by $2\pi\eta L \, dr$, we get

$$\frac{d}{dr}\left(r \, \frac{dv(r)}{dr}\right) = \frac{p}{\eta L} \, r. \tag{7.74}$$

This second-order inhomogeneous differential equation for $v(r)$ is a first-order differential equation for $V(r) = rv'(r)$ in which the prime means r-derivative

$$\frac{d}{dr} V(r) = \frac{p}{\eta L} r. \tag{7.75}$$

Since $V(0) = 0$, the solution to this equation is

$$V(r) = \int_0^r \frac{p}{\eta L} r' dr' = \frac{p}{2\eta L} \, r^2. \tag{7.76}$$

Integrating once more and applying the boundary condition $v(R) = 0$, we find

$$- \, v(r) = v(R) - v(r) = - \int_r^R v'(r') \, dr' = \int_r^R \frac{p}{2\eta L} \, r' \, dr'. \tag{7.77}$$

So the speed $v(r)$ of the fluid at radius r is

$$v(r) = \frac{p}{4\eta L} \left(R^2 - r^2\right). \tag{7.78}$$

The flow rate Q through the pipe is the integral of the speed $v(r)$ of the fluid over the circular cross-section of the pipe

$$Q = \int_0^R \frac{p}{4\eta L} \left(R^2 - r'^2\right) 2\pi r' \, dr' = \frac{\pi}{8\eta L} \, p \, R^4 \tag{7.79}$$

which is the **Hagen-Poiseuille** formula. So to the extent that blood is a newtonian fluid, the blood pressure needed for a given rate of flow varies as the inverse fourth power $1/R^4$ of the radius R of the artery.

Example 7.2 (Laminar flow)**.** Suppose we slowly stir corn syrup in which we carefully placed drops of acrylic dyes of different colors mixed with the corn syrup. The speed v is about 0.02 m/s. The length $L = 0.1$ m. The density of corn syrup is $\rho = 1.38 \times 10^3$ kg/m^3, and its viscosity is $\eta = 1.3806$ kg/m·s. So the Reynolds number for this event is

$$\mathcal{R} = \frac{vL\rho}{\eta} \approx \frac{0.02 \text{ m/s} \times 0.1 \text{ m} \times 1.38 \times 10^3 \text{ kg/m}^3}{1.38 \text{ kg/m·s}} = 2. \tag{7.80}$$

And indeed we can slowly stir the mixture backwards and recover the separated drops of colored dye. The demo is on YouTube (`https://www.youtube.com/watch?v=p08_KlTKP50`). Had we used water instead of corn syrup, the Reynolds number of the event would have been $\mathcal{R} \approx 2 \times 10^3$ which is within the range of turbulent flow. This is why one can't stir cream into one's coffee and then separate the cream by stirring it backwards. □

7.6 Fluid mechanics

The **conservation law** for a fluid of mass density ρ moving with velocity $\boldsymbol{v}$ is

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{v}) = -\rho\, \boldsymbol{\nabla} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho. \tag{7.81}$$

The total time derivative of the density then is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \rho = -\rho\, \boldsymbol{\nabla} \cdot \boldsymbol{v}. \tag{7.82}$$

An **incompressible fluid** is one for which both sides of this continuity equation vanish

$$\frac{d\rho}{dt} = 0 \quad \text{and} \quad \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{7.83}$$

The force $\boldsymbol{F}$ acting on a tiny volume dV of a fluid due to a pressure p is the integral over the surface $\boldsymbol{dA}$ of dV

$$-\oint p\, \boldsymbol{dA} = -\int \boldsymbol{\nabla} p\, dV. \tag{7.84}$$

in which $\boldsymbol{dA}$ is the outward normal to the surface. Equating this force per unit volume to the density times the acceleration, we find

$$\rho\, \frac{d\boldsymbol{v}}{dt} = \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} \right) = -\boldsymbol{\nabla} p \tag{7.85}$$

which is Euler's equation for an **ideal fluid**, that is, a fluid without viscosity or thermal conductivity. (Leonhard Euler 1707–1783)

An incompressible fluid with a constant viscosity η obeys the **Navier-Stokes equation**

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} \right) = -\boldsymbol{\nabla} p + \eta\, \boldsymbol{\nabla}^2 \boldsymbol{v}. \tag{7.86}$$

(Claude-Louis Navier 1785–1836, George Stokes 1819–1903)